THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
PROF. E. T. WHITTAKER, M.A., F.R.S.

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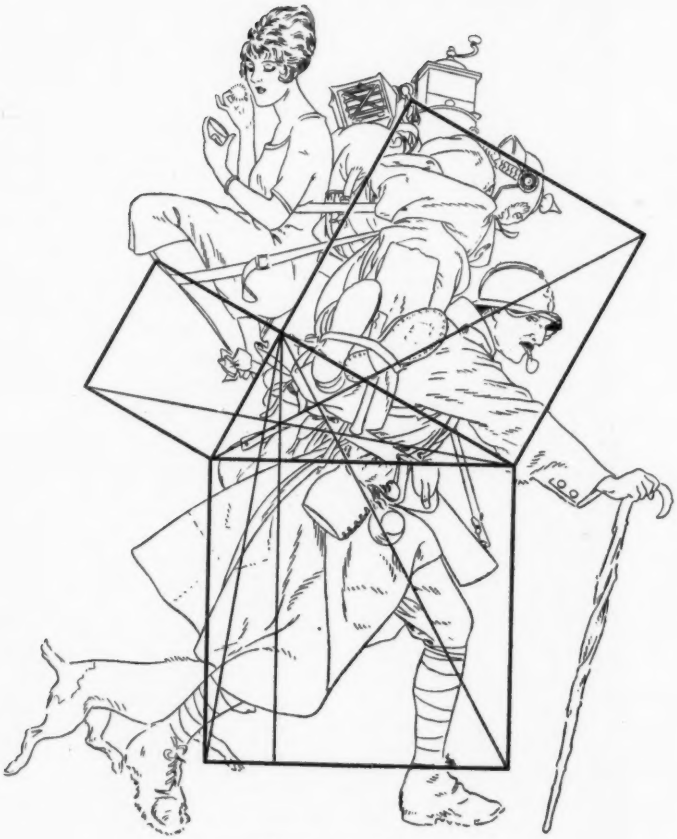

τὸ θεώρημα τῆς νύμφης.
(With apologies to *La Vie Parisienne*.)

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VOL. XI. OCTOBER, 1923. No. 166.

MATHEMATICS OF REALITY AND METAMATHEMATICS.

MATHEMATICAL ASSOCIATION (LONDON BRANCH). PRESIDENTIAL VALEDICTORY ADDRESS.

G. GREENHILL.

Feb. 31, 1923.

A VALEDICTORY Address is expected to touch lightly on some detail announced in the title, and I have chosen as not too abstruse, the subject of the Mathematics of Reality and Metamathematics.

The Science of *Applied Mathematics* may be congratulated on an escape from A Nightmare the other branches have been forced to undergo: to feel at last some slight recovery from the Tyranny of Metaphysical Rigour, converting our subject into one Aristotle might have called *Metamathematics*.

A Tyranny strewing the primrose path at the outset with boulders, brickbats, broken bottles, to retard the progress of the novice, and make him look down continually at his feet, instead of gazing up into the sky, and taking in the surrounding landscape.

According to Whitehead's Address—"different subjects and modes of study should be undertaken by the pupil at fitting times, when he has reached the proper stage of mental development"—and he attacks the principle of making the easier subject, so called, precede the harder.

The hardest task, he says, is the study of the elements of Algebra, and yet this stage must precede the comparative simplicity of the Differential Calculus.

At school the boy rises painfully from the particular to a glimpse of the general. At the University a start is made from general ideas, and the application is studied to concrete cases—concrete facts should be studied as illustrating the scope of general ideas.

The Music Master will endorse these sentiments: in his opinion Scale Practice is the great essential; the virtuoso and Algebra could not have too much of it. But the Pianola has come to the rescue of the young performer.

This style of Metaphysical Rigour seems very popular with the American mathematician, judging from the usual title of his work; all apparently made in Germany, as the programme-menus will show of their bill of fare at a Congress.

And a similar school in England is to-day of much the same provenance and tradition.

Great overlapping of Mathematics and Metaphysics in the discussion of Rigour and Relativity; so much as to keep the Vessel of Science perpetually moored in port, never to put to sea for fear of striking a Rock of Reality.

But according to John Stuart Mill, the whole of Human Activity is spent in moving Things. The Thingyness of Things stimulates the thinking man; and the man of Action too, the Engineer. And the engineer cannot afford to wait years for a little more light. He must deliver the goods.

The strategic way is to start where the last man left off, a Kelvin after Thomson, and Rayleigh after Strutt.

In Nature all the independent variables are subject to change at the same time, but in Analysis we are obliged to isolate them, and consider them to move one at a time.

We should never be able to congratulate him on his success if an investigator had taken into account all the effects neglected in a first attack.

Take the Research of Osborne Reynolds on sinuosity-stream lines in lubrication and their subsequent application in the Michell thrust block, revolutionising practice in large propelling machinery of a steam ship, unforeseen by Reynolds at the date of his experiments.

Applied Mathematics may be faulty, or split on an intellectual rock. Such work is always open to this danger. The best must be made of a hard bargain.

We arrive at nothing if we attempt to take into account every side track. But it helps the progress to establish some points about which doubt existed previously.

We should make it our object to consider this World the finest place ever we were in. To investigate the Laws of Nature and the secrets of their working, and to convert them to our best advantage.

Here we are led on to the Cosmical Theories of the Cosmogony, fascinating to the young school of *Applied Mathematics.*

Rigour would make himself more useful and acceptable to us, if he would condescend occasionally to show how we are right in our conclusions, when he declares we have no right to be right. And at the same time point out how another conclusion would finish, and *why.* Explaining to us at some length why the thing is *not so*, at the same length as to show when it is.

In our new progress we should strive to make the path smooth and easy, and one important way is to exterminate the needless approximation, unacceptable to the young Berkeley mind as *stilted, insidious, insincere*; where it is quite as easy and more convincing to give an exact treatment, to make the argument *flawless, convincing,* and *complete.*

Take for example a discussion of the small oscillation about a position of equilibrium, required for the assurance of the stability of a System, such as the Solar System of Laplace.

It will generally be found that the exact equations are quite as simple for a general motion in a penultimate state, adjacent and beyond the position of Equilibrium; and these equations require a mere return to the special case of small invisible oscillation to give an exact result.

The oscillation of a pendulum, invisible or visible, is the type of such cases, and will serve as our illustration.

Twenty years ago I wrote an article for the *Nouvelles Annales*, "*le pendule simple sans approximation*," where the treatment was purely geometrical. The article has attracted no subsequent notice, so we may return to the well known differential equation of motion of a simple pendulum of length l, in finite oscillation, (1) $\frac{dQ}{dt}+\frac{g}{l}\sin\theta=0$, with $Q=\frac{d\theta}{dt}$.

We do not seek here to integrate this equation, and so be led into the mazes of the *Elliptic Function*.

But differentiating again, with $Q\,dt = d\theta$,

$$\frac{1}{Q}\frac{d^2Q}{dt^2} + \frac{g}{l}\cos\theta = 0,$$

an exact equation still.

In the invisible evanescent oscillation considered in an elementary treatise, $\theta = 0$, and $\frac{1}{Q}\frac{d^2Q}{dt^2} + n^2 = 0$, with $g = ln^2$, and here is the Differential Equation, from which there is no escape, of simple vibration, usually called *harmonic*. Better follow Schuster's advice and drop harmonic, as otiose and misleading: call it *Simple Vibration* for short. The word Harmonic was introduced in Thomson and Tait at a time when the musical vibration of a cord or organ pipe was principally under consideration as the type.

In an elementary course, before the study is taken up of the Differential Equation, the name at least may be assumed as familiar, and the solution taken for granted, verified immediately by differentiation as developed by Perry in his popular lectures on *Practical Mathematics*.

Make a start on $\theta = A[\cos nt$, or $\sin nt$, or $\cos(nt + \epsilon)]$, introducing a lead or lag ϵ, to allow for phase difference, representing a vibration with a beat given by $nt = \pi$, $t = \frac{\pi}{n}$ or a double beat-period by $nt = 2\pi$, $t = \frac{2\pi}{n}$; a frequency $\frac{n}{\pi}$ beats per second.

Beat and period must be kept constantly under observation for the treachery of the factor 2; the vibration is shown in a shadow on the wall of the uniform motion in a horizontal circle of the bob of a conical pendulum (show this).

The period of revolution round the circle gives two beats, from rest to rest of the shadow, imitating the motion of a plane pendulum; and here is a simple demonstration.

The seconds pendulum beats one second, but a conical pendulum of the same height, about one metre, has a two-second period.

The contrast is seen in the exact treatment of the pendulum by the Elliptic Function, either oscillating through a finite visible angle as a clock pendulum, or in the associated motion of complete revolution, as of a church bell in a peal, contrasted with the tolling of the tenor bell.

While the centre of oscillation P of a clock pendulum makes a visible beat from B to B', a point Q on the associated church bell can accompany P at the same level, round a circle in a complete revolution. The relation between the S.E.P. l of the tolling bell P, swinging through an angle 2α, and the associated pealing bell Q is a little delicate question in Elliptic Function Theory.

By the introduction of the Elliptic Function all trace of approximation has been obliterated, and there is no flaw in the argument to attract the polite critical attention of Rigour.

In the penultimate case above, we have preferred to work to Q, the velocity of the vibration and its fluctuation, instead of the displacement. This method is useful in the discussion of very rapid vibration, as of sound, electricity, and Light-Diffraction. The number representing the velocity is n times that of the displacement a, and n the frequency is a large number in such cases, so that na may be appreciable, while a is invisible.

The problem of Tube Collapse will serve as another application of the method just employed, of obliterating approximation in the treatment of the pendulum in small invisible oscillation, and need not be more complicated.

Considering a finite arc AP of the cross section of any shape of a cylindrical tube exposed to uniform external pressure X, a simple application of statical principles (*Math. Ann.*, 1898, Engineering, March 15, 1918) will show that the difference of the bending moment (B.M.) at the ends of the arc AP is $\frac{1}{2}X(OA^2 - OP^2)$ per unit axial length, where O is a fixed origin.

Assuming that the B.M. produces a proportional change of curvatures $\frac{1}{\rho}$, we arrive at Lévy's relation of this Elastic

$$\frac{1}{\rho} = 4Ar^2 + 2B,$$

where $8A$ denotes the quotient of the pressure X divided by the flexural rigidity, $EKk^2 = \frac{1}{12}Et^3$, for thickness t inches, with E the modulus of elasticity, measured in lb/inch^2, as also the pressure X.

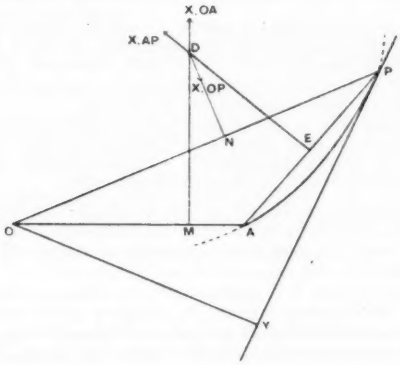


FIG. 1.

We work to the inch and pound units, employed by the engineer, as his physical constants are given in this system; and then taking any finite arc AP of the cross section and considering the equilibrium of an inch run of axial length, the resultant pressure thrust on the arc AP is the same as across the chord AP and is $X.AP$ acting at right angles to the chord and through its mid-point E.

Draw the line AO at right angles to the resultant stress at A, and such that $X.AO$ is the magnitude of the stress; this will not act through A, but along DM suppose, in consequence of the bending moment at A, $X.OA.AM$. If this force crosses the line of pressure thrust ED in D, then by the Triangle of Force, the stress at P must be $X.OP$, along DN perpendicular to OP, with a bending moment $X.OP.PN$ at P.

If the plate is bent from the flat to a radius of curvature ρ at P and ρ_0 at A, the elastic resilience of the bending at P and A,

$$\frac{EI}{\rho} = X.OP.PN = \tfrac{1}{2}X(OP^2 + PD^2 - OD^2),$$

$$\frac{EI}{\rho_0} = X.OA.AN = \tfrac{1}{2}X(OA^2 + AD^2 - OD^2),$$

and their difference $$\frac{EI}{\rho} - \frac{EI}{\rho_0} = \tfrac{1}{2}X(OP^2 - OA^2),$$

and with $\frac{X}{EI}$ replaced by $8A$, and $OP = r$, this leads to the form above,

$$\frac{1}{\rho} = \frac{dp}{r\,dr} = 4Ar^2 + 2B,$$

and integrating $$p = Ar^4 + Br^2 + C.$$

These 2, 4, 8 factors were introduced at the outset of the work with the object of arriving at this simple form of the (p, r) relation.

In every curve of cross section AP, with the arc $AP = s$, $PY = q$, $OY = p$,

$$q = r\frac{dr}{ds}, \quad \frac{1}{\rho} = \frac{dp}{r\,dr} = \frac{dp}{q\,ds}, \quad q^2 = r^2 - p^2, \quad q\frac{dq}{ds} = r\frac{dr}{ds} - p\frac{dp}{ds},$$

$$\frac{dq}{ds} = 1 - p\frac{dp}{q\,ds} = 1 - \frac{p}{\rho}, \quad \frac{d^2q}{q\,ds^2} = \frac{d}{r\,dr}\left(1 - \frac{p}{r}\right) = -\frac{1}{\rho^2} - \frac{p}{r}\frac{d}{dr}\left(\frac{1}{\rho}\right).$$

Applied to the equation of Tube Equilibrium,

$$\frac{1}{\rho} = 4Ar^2 + 2B, \quad \frac{d}{dr}\left(\frac{1}{\rho}\right) = 8Ar, \quad \frac{d^2q}{q\,ds^2} = -\frac{1}{\rho^2} - 8Ap,$$

an exact equation, interesting illustration of differentiation; and if of circular cross section, radius a, with $s = a\theta$, $\rho = a$,

$$\frac{d^2q}{q\,d\theta^2} = -1 - 8Aa^3 = -n^2, \quad n^2 - 1 = 8Aa^2;$$

in the invisible deviation from a circular form, with a solution

$$q = Q\sin n\theta = Q\sin n\frac{s}{a},$$

and n waves round the circumference.

Here in this treatment the word "approximate" has been banished, exterminated, eliminated (filed off). A search should be organised to hunt down similar cases.

For the measurement of time in a dynamical experiment, a simple equivalent pendulum (S.E.P.) should be employed, a thread and plummet, and the length l of thread noted to give a beat of coincidence with an oscillation.

No clock, watch, timepiece should be used in this the method invented by Galileo for his lectures, suggested by the swinging lamp in Pisa cathedral.

I have a portrait, suitable to serve as a vignette for a treatise.

And it is noticeable in the Principia how g does not make an appearance explicitly, but the length L instead of the S.E.P. to beat the second, making $g = \pi^2 L$.

The usual pendulum formula of the S.E.P. beat, $\pi\sqrt{\frac{l}{g}}$, is then replaced by the simpler form $\sqrt{\frac{l}{L}}$, seconds.

In the record of an oscillating system, the S.E.P. l is calculated, and should be tabulated as a geometrical length, instead of the time of a beat, $\pi\sqrt{\frac{l}{g}}$, $= \sqrt{\frac{l}{L}}$.

The title *Differential Equations* (D.E.) is held up as a type of an advanced difficult subject, not to be attempted till after a thorough course of the Calculus—*Integral and Differential* (I.C.) and (D.C.). But I would introduce the name and idea at the outset, in accordance with Whitehead's advice to start from general ideas.

Nature talks to us in the Differential Equation to express her Laws. The constant is inserted to suit the special concrete case in hand.

So instead of the bald demand of the Integral Calculus of—"Integrate this function of x, $f(x)$"—it will accustom the novice to the terrifying name of the thing if the request is made in a slightly different form, quite as polite—"Solve the D.E. $\frac{dy}{dx} = f(x)$," where the variables are separated and give a graph of the solution, showing the effect of the arbitrary constant C, in

$$y + C = \int f(x)\,dx,$$

explaining the origin of the sacred symbols, long $\int$, and little d.

The elimination of the constant by a differentiation brings back the D.E.

And generally the idea of *Integration* is easier to grasp in a mental picture as more tangible than a corresponding *Differentiation.*

A return after years will show the growth of the trees of a park, as the Integration in the interval. But the actual rate of growth is invisible, as expression of the D.C.

A journey of given length is performed in a stated time, as on a railway. But the variation of velocity is a secondary consideration, and can be estimated only roughly by timing the quarter mile posts.

So I would change the order of the title to "Integral and Differential Calculus," and set the beginner to work in a *viva voce* class, as if reciting the Multiplication Table, in a start of an "integrate 0, 1, x, x^2, x^3, ..., and then interpret by the average value."

The average value of x^n from 0 to x is $x^n/(n+1)$, and so of x is $\frac{1}{2}x$, and so on.

The class gain confidence if they find they are right at the outset in their guess.

Then they may be set to work on a restatement, and asked to solve the D.E.

$$\frac{dy}{dx}=0,\ 1,\ x,\ x^2,\ \dots\ .$$

Take the D.E. $\quad \dfrac{dy}{dx}=\dfrac{1}{\sqrt{x}},\quad \sqrt{x},\quad \dfrac{1}{\sqrt{x}}\pm\sqrt{x},\ \dots$

and their solution $\quad y+C=2\sqrt{x},\quad \frac{2}{3}x^{\frac{3}{2}},\quad 2\sqrt{x}\pm\frac{2}{3}x^{\frac{3}{2}},\ \dots.$

The Singular Solution is seen here graphically, as a tac-locus, cusp-locus, node-locus, or string of conjugate points, and the name of these things need have no future terrors.

Perry was an advocate of this method, and made it a success with his large class of elementary students, never likely to have future time to go much deeper into the extensions.

And F. G. Hall has developed the same idea in his discussion in the Jan. *Math. Gazette* on *Pure Mathematics in a School Advanced Course.*

In my own case I feel how I should have saved a year of bewilderment if I had been introduced at the outset to the notion, in mere mention, of $\dfrac{dx}{dt}$ as a velocity, in the dynamical interpretation of a D.C.

And familiar statements will make a direct appeal, such as

one % increase in x gives n% to x^n;

or $\quad m$ % increase in x^m gives n% to x^n;

interpretation of $\quad \dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{n}{x}\quad$ or $\quad \dfrac{m}{y}\dfrac{dy}{dx}=\dfrac{n}{x}$,

in $\quad y=x^n\quad$ or $\quad y^m=x^m$.

The integration of $\sin x$ or $\cos x$ may be exhibited in the same way, and without requiring any D.C.

Take for example the curve, graph of $y=\cos x$, Fig. 2, to the same lettering as before.

With the circular arc $BQ=NP=x$, $qq'=mm_1$, and with $PM=QH$,

$$\mathrm{Lt}\,\frac{hh'}{qq'}=\frac{QH}{OQ}=\frac{PM}{OB},$$

$$\mathrm{Lt}\,\frac{\text{trapezium } pm_1}{\text{trapezium } hk'}=\mathrm{Lt}\,\frac{PM\,.\,mm_1}{OB\,.\,hk'}=1.$$

Then as p, p' separate from P and q, q' from Q, this equality still persists, so that the finite area

$$P_2M_1 = \int_{x_2}^{x_1} \cos x\, dx = \text{rectangle } H_2K_1$$
$$= \sin x_1 - \sin x_2.$$

The same method can be illustrated with the integration of $\sin x$.

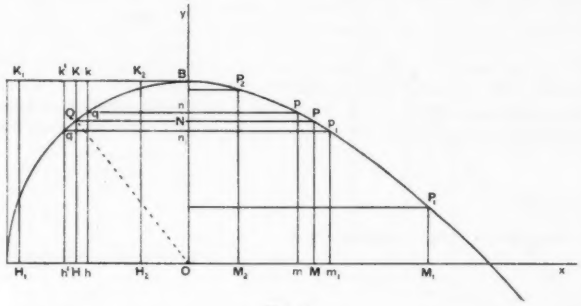


FIG. 2.

Search for other illustrations showing to the eye a picture of an integration and its inner meaning. Take for instance the determination of the C.G. of the area P_2M_1, the gradient to Oy is one in n, one horizontal to n vertical (or m horizontal to n vertical).

Thus for a graphical representation of the integration $\int y\, dx$, with $y = x^n$, or $x^{\frac{n}{m}}$ for illustration.

Draw a tangent PT to the curve $y = f(x)$ at P; the gradient to Oy will be one in $\frac{dy}{dx}$, or one in $\frac{dx}{dy}$ to Ox, the usual horizontal.

A small step Pp back, or Pp' forward, and a trapezium pm' and pn' on each side of P, with average height PM or PN, making up a gnomon $mp'n$.

If TQ is drawn parallel to Oy, Euclid I. 43 may be restated as—the complementary trapeziums pm', pq', about the diagonal PT are equal.

By taking a step back as well as forward, the equivalent of running through P, without a stop and past the limit, not coming to a dead stop there, a certain grittiness is avoided in the usual statement, making it more to the taste of a young Berkeley.

If the curve is $y = x^n$ $\left(\text{or } x^{\frac{n}{m}}\right)$ the D.C. asserts that one % advance in x gives n %, or $\left(\frac{n}{m}\right)$ % increase in y.

Then $NP = n \,.\, QP$, or the trapezium pn' is n times the trapezium pq' or pn'.

This constant ratio holds all along the curve $y = x^n$, so that if p, p' separate from P along the curve to a finite distance, at P_1 and P_2, the curve divides the gnomon $M_2P_1N_2$ in this number ratio.

The area of the gnomon, difference of the rectangles, is $x_1y_1 - x_2y_2$; so that the area P_1N_1, $P_2N_2 = \frac{1, n}{n+1}(x_1y_1 - x_2y_2)$.

Here we see the meaning of Integration by Parts in the formula of the I.C.

$$\int y\, dx + \int x\, dy = xy + C.$$

Differentiation by Parts is the rule for a Product,

$$\frac{dyz}{dx}=\frac{dy}{dx}z+y\frac{dz}{dx}.$$

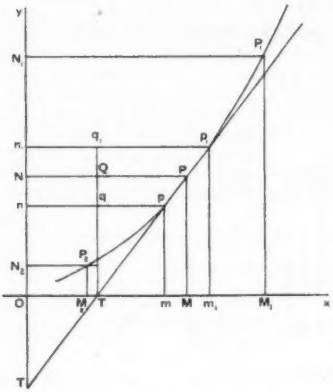


FIG. 3.

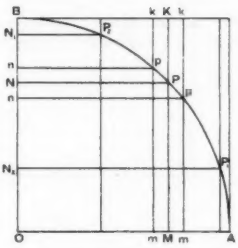


FIG. 4.

Archimedes employs the method in his Cubature of the Sphere and Quadrature of its surface; and so he may be considered Inventor of the Integral Calculus.

In a revolution of the quadrant AB in Fig. 4 about the radius OA, the annular area swept out by PK is equal to the circle on the radius PN, swept out by revolution round OB.

The elements of surface swept out by pp', kk' are equal, and the surface of the sphere is thus equal of the curved surface of the cylinder.

The ratio of the two volume elements, swept out by kp' about OA and np' about OB, is then

$$\frac{\pi x^2 dx}{2\pi xy\,dy}=\frac{\pi x}{2\pi x}=\frac{1}{2}, \quad \text{with} \quad x\,dx=y\,dy,$$

so that the sphere divides the circumscribing cylinders as $2:1$; or the sphere is $\frac{2}{3}$ volume of the cylinder.

I ought to excuse myself for bringing forward such trite elements before this erudite company, but I wished to point out how a certain amount of *grit* may be removed in the usual argument, making the treatment more acceptable to young Berkeley.

At the same time I did not recommend these considerations to his attention in his incipient stage, where his progress will be made chiefly by the use of a memory for formulas before learning the proof, as he was required to learn the Multiplication Table by heart and to use it.

It would be too cruel to pull him up at the outset to make him give a Rigorous Proof that Twice Two is Four, and nothing else.

In the exponential curve, the rate per cent of increase is constant, and the function grows at Compound Interest.

The idea should be introduced at the start, by the definition of the Antilogarithm of x, $10^x=\text{al}\,x$, say y, instead of the usual inverse function, $x=\log_{10}y$; and no other base should be mentioned.

Then as Perry showed, it is an easy exercise to set out the graduations of the Slide Rule by ordinary arithmetic, extraction of the square root, the cube, the fifth root of 10, and to tabulate them as an antilog table to 3 figures.

Our President of the Mathematical Association, Sir Thomas Heath, has carried out a great work in his historical treatises on *Greek Mathematics*, and I predict his little volume of Euclid, First Book in Greek, will have far reaching influence in restoring a study of the Greek language, threatened with extinction lately. Ideal book as a school prize, in showing how the treatment of Euclid in the original Greek is word for word the same as he has been accustomed to find in the usual version of Potts, or other modern equivalents.

I have dared to propose as a diagram a tracing of a cartoon in the French journal *La Vie Parisienne*, as an accidental illustration of the theorem of Pythagoras, Euclid I. 47, and of Plato's name for it, τὸ θεώρημα τῆς νύμφης: the bride borne up aloft in the procession on the θρόνος, chair of honour.

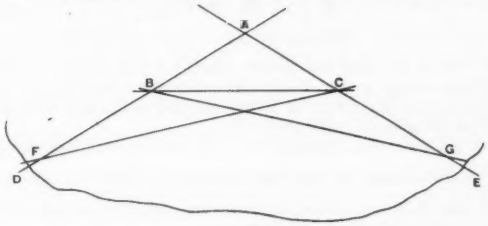


FIG. 5.

In a modern version we find it as an inn-sign—"The Man laden with Mischief," in French "sac à malice."

And the name Pons asinorum was not conferred on Euclid I. 5 for any great difficulty, as a stumbling block, but because the figure is a representation of a military trestle bridge (Fig. 5) with a ramp too steep for a horse to tackle;

a pons, non equorum, sed asinorum.

The figure is drawn usually as steep as a church roof, and then this interpretation is not so manifest.

But all these historical legends disappear with the abolition of Euclid.

I am tempted here to wander off again, into a reminiscence of the British Association in Canada. We were promised a visit to Niagara, crossing Lake Ontario from Toronto. As this involved touching at an American port, the stern Custom House difficulties of the United States were overcome by allowing us to carry a High Officer of theirs to perform his ceremonies on board.

His Majesty was in a very good humour, and approached a member of our Association, with a professional joke never known to fail on the Cunarder entering New York—"Well, my man—? Got any old clothes here?"

Our member invited him to look for himself, as his wallet was open. But His M. merely chalked his mark, and passed on, and this made our member exclaim with astonishment—"Why, he has chalked the mystic symbol, the Mystic Pentagram of Pythagoras."

This was too much for the son of the Custom House officer, typical sharp American boy, brought by his father to watch and hear all the strange people around. He felt compelled to cut in—

"No, Sir! that, Sir, is The Star of The United States."

The Mystic Pentagram is an interesting exercise in *Paper Folding*, tying a knot in a ribbon, and flattening it out.

Twist the ribbon to form the unifacial surface, and the hexagon is obtained.

Other illustrations in *La Vie Parisienne* of mediaeval head gear may be employed as a diagram in Paper Folding.

Invited some years ago to deliver an Address to the Mathematical Club at Cambridge, I chose as the title—A Lecture of one minute on Linear Dynamics. Starting on Newton's Definitions and Laws of Motion, where Quantity of Motion, or Momentum, is defined as product, Wv, of Quantity of Matter, W lb, and velocity f/s, the Law states that change of Momentum is proportional to the Force F, lb, in gravitation measure, the only measure in existence in Newton's day and 200 years after, and still in universal use among Engineers. This change requires to be qualified as Time Change, so that if the momentum grows from rest in t seconds, Wv and Ft are proportional on Newton's Law; double time will add double velocity and momentum.

To convert the Proportionality of Newton's statement in the Principia into the Equality of our day, the Law must be written

$$W\frac{v}{g} = Ft \quad \text{(lb-sec)},$$

(lb) (sec) (lb sec)

requiring the factor g; and the proper place for g is under v; it must not straggle and take cover under W, in the old-fashioned confusing way, when W/g was replaced by the letter M in the old-fashioned lazy way, and M was called the *Mass* sui generis, with unit g lb.

But gravity makes no such distinction. All matter is transparent to gravity.

Because $\frac{v}{g}$ is the seconds of free fall to acquire velocity v, and corresponds to t, while W and F are measured in the same unit, the pound defined in the Act of Parliament on Weights and Measures.

Thence from the average velocity, taken as the arithmetic mean (A.M.),

$$\frac{1}{2}v = \frac{s}{t},$$

multiplication gives $$W\frac{v^2}{2g} = Fs, \quad \text{(lb-ft or ft-lb)},$$

(lb) (feet) (lb feet)

so that double velocity requires four fold distance s or force F, and $\frac{v^2}{2g}$ are the feet free fall from rest to acquire the velocity v; diluted in the equation by the factor $F : W$; as on an incline for instance.

Here is Dynamical Principle enough to carry us a long way in the ordinary questions that meet the eye on the road, in the traffic, in the workshop working with tools.

The Hammer—Nail—block experiment can be shown off in illustration.

For the block to recoil freely without rotation it can be slung by threads on the 5-point principle (Right as a Trivet).

And More to it. Encourage the class to look up off the page and out of the window at the dynamical display to be seen everywhere around.

The daily experience of Speed sighting, in threading the crowd of traffic, may serve as an introduction to vector Composition of Velocity.

The Engineer is complaining to-day of another tyranny, of the C.G.S. system, insisted on by the pedantic professor in his lectures, to harmonize with the Electrical interest, of volt, ohm, ampère, farad, erg, joule, watt.

But our Engineer will continue to record his calculations in the pound or ton, inch or foot, of Imperial Standards of the Act of Parliament.

Or his continental rival will work to the kilogramme and metre.

Such niggling units as *centimetre* and *gram* were selected for the fancied simplification of making water density unity, and number for density and specific gravity the same number.

On the M.K.S. (metre-kilogramme-second) system, the density of water is 1000, kg/m^3, with the advantage of recording other densities as four figure results, rarely going into a decimal; unless air density is taken into account in a very accurate weighing, requiring a subtraction of about 1·5 from a recorded number of absolute density, in vacuo, as in the definition of our pound in the Act of Parliament, and in the determination of the volume in cubic inches of the gallon, of liquid measure: 277·3 to the gallon 10 lb of water, and here is the familiar 27·73 of the *gunner*, 27·73 inch^3/lb.

And aeronautical calculation depends essentially on air density.

Our standards are defined at a comfortable temperature of 62° F, in a normal atmosphere of a barometric height of 30 inches, agreeable conditions to work under in a long careful experiment.

But the pedants of the Metric System, with merciless logical rigour, took a standard temperature of 0° C, the freezing point. As a consequence a comparison with the Standard is carried out in Arctic conditions, and a temptation arises to hurry over the work.

Elementary Mechanics are inculcated in school to-day through C.G.S. units, never encountered in the affairs of life, and contact is lost with the reality seen around.

A desideratum is a Text Book of Elementary Mechanics excluding Electrical Interest, with a return to an exclusive use of our Imperial Measures, pound and foot, and gravitation measure; with a pendulum of S.E.P. l to mark out equal intervals of time.

Galileo's discovery of the pendulum meant his appreciation of the simple pendulum for this purpose in his dynamical experiments, in place of the Clepsydra of his day. So easy to verify with trucks on an inclined plane his laws—

de motu gravium naturaliter accelerato—

how four fold distance is described in two beats, or in one beat of a pendulum of four fold length.

Replace the two clocks of Relativity by two pendulums.

It is time for his jibe at Aristotle to be discounted; this brought Galileo into conflict with the Inquisition, and not any theological heresy.

Because Aristotle in his Physica-Mechanica is discussing the terminal velocity in a resisting medium, as of a rain drop or hail stone, or the rise of a bubble in water.

The Inquisitor could disregard any theories of Galileo on theological science as beyond his powers in heretical doctrines on the *Mass*, but would not overlook any attack on the Infallibility of Aristotle.

I have a postcard somewhere, but I have mislaid it, of the swinging lamp at Pisa: good for a vignette of an Elementary Dynamics to add to the Leaning Tower of Pisa.

I invite attention here to these diagrams* of exact *Gyroscopic Motion* drawn by Sir George Hadcock on calculations made by Colonel Hippisley, shown on the wall.

G. GREENHILL.

GLEANINGS FAR AND NEAR.

181. Mathematics are perhaps too much studied at our Universities. This seems a science to which the meanest intellects are equal. I forget who it is that said: "All men might understand mathematics if they would."—Goldsmith, *Essay on Polite Literature.*

182. Sir Thomas Staples used to say that he became the surviving member of the Irish House of Commons on the death of Quentin Dick. He had gone the north-west circuit with Quentin Dick who owned a gig. "You may suppose our journey was not very comfortable, as his wheels were polygons."

* Published since in the *Phil. Mag.*, Sept. 1923.

A SUPERIOR LIMIT TO n IN FERMAT'S EQUATION $x^n+y^n=z^n$ FOR A GIVEN VALUE OF z.

By Paul R. Heyl, Bureau of Standards, Washington, D.C., U.S.A.

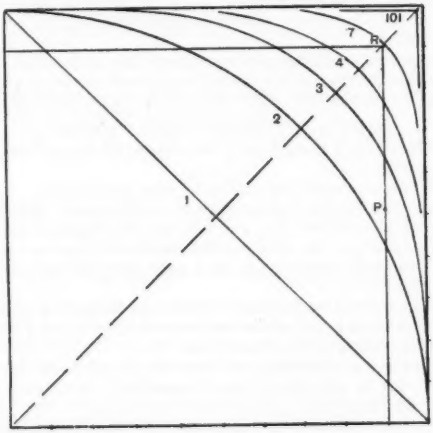
If in Fermat's equation $z^n=x^n+y^n$ .....(1)

we regard z as a given constant, and, assigning different integral values to n, plot the resulting curves, we obtain a family of curves such as is shown in the accompanying figure, which is constructed for $z=10$. When $n=1$ the locus is the straight line $x+y=10$.

When $n=2$, we have the circle

$$x^2+y^2=10^2.$$

For larger values of n the successive curves of the family approach as a limit the upper and right sides of a square of side 10, whose two remaining sides coincide with the axes of coordinates. As n increases, the successive curves are more and more closely spaced. That this is the case may be seen from the following considerations.



At the apex of any curve (its intersection with the dotted line in the figure) $x=y$, and consequently $z^n=2x^n$,

$$x=\frac{z}{2^{1/n}}.$$

As n increases indefinitely, x approaches the limit z by smaller and smaller steps.

Differentiating Fermat's equation, regarding z as a constant,

$$y^n=\text{constant}-x^n,$$

$$\frac{dy}{dx}=-\left(\frac{x}{y}\right)^{n-1}.$$

For any point of the locus above the dotted line, x is less than y, and as n

increases indefinitely, $\frac{dy}{dx}$ approaches zero from the negative side. Similarly, for any point below the dotted line, x is greater than y, and $\frac{dy}{dx}$ approaches infinity as n increases.

This family of curves can, of course, be extended to include negative values of n, x and y, with results which are of no interest in this connection, as Fermat's problem is historically limited to positive integral values of the four quantities concerned.

Construct the square of side 9, with its left and lower sides coincident with those of the larger square. Nearly all the curves of the system will lie entirely in the space of unit width between the two squares. The first curve to lie entirely in this space will apparently be that for $n=7$. For all points on this curve, and all higher curves, either x or y must differ from z by less than unity; and hence for all values of n above a certain critical value, there can be no integral solution of Fermat's equation.

This critical value of n is a function of z defined in general by the equation of the curve with its apex at the point $x=y=z-1$.

$$z^n=2(z-1)^n, \quad \text{.....(2)}$$

$$n=\frac{\log 2}{\log\left(\frac{z}{z-1}\right)}.$$

The following table gives a few values of n:

z	Critical value of n	$\frac{n}{z}$
1	0	0
2	1	0·5
3	1·71	0·570
4	2·41	0·602
5	3·11	0·621
10	6·58	0·658
100	68·92	0·689
1000	692·80	0·693

It will be seen that the ratio $\frac{n}{z}$ apparently approaches a limit as z increases. This limit, in fact, is $\log 2=0{\cdot}693$, as is seen by the following:

$$n=\frac{\log 2}{\log\left(\frac{z}{z-1}\right)}=\frac{\log 2}{\log\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots\right)}.$$

Let z increase indefinitely, and we have for the denominator

$$\log\left(1+\frac{1}{z}\right)=\frac{1}{z}-\left(\frac{1}{2}\cdot\frac{1}{z^2}\right)+\left(\frac{1}{3}\cdot\frac{1}{z^3}\right)\ldots.$$

Hence, in the limit, $n=z\log 2$.

As may be seen by the table, for $z=10$ the first integral value of n beyond the smaller square is 7. For this and all higher values, no integral solution exists for the equation $10^n=x^n+y^n$.

For $z=100$, no integral solutions are possible if $n=69$ or more; and for $z=1000$, for $n=693$ or higher.

As a matter of fact, the point R, when $x=y=z-1$ cannot itself furnish an integral solution of (1) for any value of z, for in such a case we would have, by (2),

$$\left(\frac{z}{z-1}\right)^n=2,$$

which cannot be true in integers. It is easily seen that for all values of z, z and $z-1$ are relative primes; hence their nth powers are also relative primes, and their ratio can never equal an integer. The first point for which an integral solution of (1) may be possible will be some other point, one or both of whose coordinate differences $(z-x)$ and $(z-y)$ are greater than unity. Such a point, as P, will always exclude a greater number of members of the family of curves than the point R; hence the superior limit to n set by R should be capable of being considerably reduced.

What point will offer the first possibility of an integral solution? We suppose that n is a prime number.

Write equation (1) in the form

$$z^n=(z-l)^n+(z-m)^n. \quad \ldots\ldots(3)$$

It is known that if (3) is to be solved in integers, l and m must be limited to certain forms. There are two cases to be considered:

(a) l and m are both prime to n.

(b) One of them, say l, is divisible by n.

In case (a) we must have

$$\left.\begin{aligned} l&=a^n,\\ m&=b^n,\end{aligned}\right\} \quad \ldots\ldots(4)$$

where a, b and n are relative primes.

In case (b) we must have

$$\left.\begin{aligned} l&=a^n n^{n-1},\\ m&=b^n,\end{aligned}\right\}. \quad \ldots\ldots(5)$$

where a, b and n are relative primes. These conditions were given by Barlow * amongst others.

The smallest possible values of l and m are those obtained by putting $a=1$, $b=2$ in (4), since, when n is greater than 2, 2^n is always less than n^{n-1}. Such a point is represented by P. It will lie on the line $x=z-1$, and its ordinate will depend upon the value of z for which the figure is drawn. Substituting $l=1$, $m=2^n$ in equation (3), we have

$$z^n=(z-1)^n+(z-2^n)^n \quad \ldots\ldots(6)$$

or

$$1=\left(1-\frac{1}{z}\right)^n+\left(1-\frac{2^n}{z}\right)^n. \quad \ldots\ldots(7)$$

Equation (7) is not readily put into a form in which n is expressed as an explicit function of z. If we assume that $\left(1-\frac{1}{z}\right)^n=1$ we obtain $n=\frac{\log z}{\log 2}$. The following table gives the correct values of n, obtained by computation, and the above approximate value:

z	n	$\frac{\log z}{\log 2}$	Difference
10	2·29	3·32	1·03
10^2	5·39	6·64	1·25
10^3	8·71	9·96	1·25
10^4	12·06	13·29	1·23
10^5	15·41	16·61	1·20
10^{10}	32·09	33·22	1·13
10^{100}	331·17	332·19	1·02
10^{1000}	3320·93	3321·93	1·00
10^{10000}	33218·3	33219·3	1·0
10^{100000}	332192	332193	1

*Barlow, *Theory of Numbers*, London, 1811, pp. 160-169, or L. J. Mordell, *Three Lectures on Fermat's Last Theorem*, Cambridge, 1921, p. 27.

Apparently, the limit approached by n as z increases is

$$\frac{\log z}{\log 2} - 1.$$

The value of n in the second column being an increasing function of z (although very slowly increasing), it is therefore impossible to arrive in this way at a general proof of Fermat's Last Theorem. If any superior limit could be set which is constant, or a decreasing function of z, it would be only a matter of time and labour to work upward after the manner of Kummer and others until this limit was reached.

Since when $z = 10$, $n = 2{\cdot}2909$, it follows that $2^n = 4{\cdot}8936$. In this case the first point P for which an integral solution is possible has the coordinates $x = 9$, $y = 5{\cdot}1064$ as represented in the figure. All integral values of n greater than 2 are excluded, agreeing with the table, which sets the critical value of n in this case between 2 and 3. P. R. HEYL.

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183. [At Paris.] A crowd of people came in the evening to hear the news —amongst others, La Place, the savant. He was so civil as to recognise me as a *colleague*, because I was an F.R.S. God knows how he knew that; we got into conversation; his was certainly not brilliant, nor was there, I ought to add, any opportunity for brilliancy, but a little trait of character escaped him. We talked of the King's administration before the *Cent Jours*. He did not approve of the tone which had been adopted and still less of the measures, but there was one thing which was so gross a blunder and folly that it exceeded all the rest, and was indeed the chief cause of the return of Buonaparte. I pricked up the ears of curiosity at this exordium, and felt no small interest to hear what this portentous blunder had been. It was, that instead of making Buonaparte's senators all *hereditary peers*, some of them had the peerage for the same term they had their senatorship only, viz. *for life*. M. le Comte de La Place was only a *peer for life*. The King, however, soon adopted this new system, and made all the peers hereditary, and never did a more unwise thing. La Place, for instance, is as fit to be a peer of France as he is to be a drum-major of a regiment of the line. He was, however, a respectable man, and the thing was tolerable in his personal case, but it is different when his sons and all the La Places to the end of the chapter must be the colleagues, not of F.R.S.'s like me, but of the Noailles, Montmorencies, and Birons. A peerage without either gold or blood!—Croker's *Note Books*, Oct. 26, 1826.

184. The number of all possible arrangements of letters, repeated or not, and capable of being pronounced or not, up to words of 24 letters, is of the following order of magnitude. Take a million of millions: repeat it a million of million times: a result is between 1391 and 1392 millions of such numbers. As an instance of the manner in which the dropping of consonants and confusion of vowels may permit possible alteration of spelling, M. de Mairan computed that the word *Hainaut* might be spelled in 2304 different ways, so as to be pronounced in the same way by as many different Frenchmen, or very nearly so.

The most useful proposition in the higher part of the theory of combinations is the reduction of the formula $1 \cdot 2 \cdot 3 \ldots (x-1)x$ to a very close approximation, which can be easily calculated by logarithms. It affords, at the same time, a useful lesson to those who have not studied mathematics at all, or very little; we have seen ignorance comfort itself with laughter more than once at the idea of the preceding product being found by employing the proportion which the circumference of a circle bears to its diameter.—De Morgan, [*Combinations and Permutations, P.C.*, 1837].

MATHEMATICAL ABILITY—A PLEA FOR RESEARCH.

1. It is remarkable that, although mathematics holds an unchallenged position as part of the school curriculum, and mathematical subjects form part of most public examinations, so few attempts have been made to analyse mathematical ability. The importance of such analysis is no less than its difficulty, and the research that is waiting to be undertaken is as interesting as it is wide. This article is written in order to point out some of the problems awaiting solution, and to invite further discussion and the co-operation of members in the proposed inquiry. Without such co-operation little can be achieved, for statistical inquiries are not to be undertaken single-handed, either in the collection or in the mathematical treatment of data.

2. Almost the only inquiry into mathematical abilities and their mutual relationships hitherto published in this country is Dr. Wm. Brown's paper on "An Objective Study of Mathematical Intelligence," which appeared in the volume of *Biometrika* for the year 1909. The data of that paper are, however, based on the study of only eighty-three school pupils, and Dr. Brown refrains from drawing more than a few tentative conclusions. A paper on the correlation of mathematical abilities, published this year in Vol. I. of *The Forum of Education*, is the only other attempt to cover the same ground as Dr. Brown's paper, and is unsatisfactory for several reasons that we need not state here. The relevant work of Burt, Ballard, and Winch consists of studies of various aspects of arithmetical ability, but does not cover the wider field of school mathematics.

In America, as in England, investigations into arithmetical abilities are much more numerous than those into other branches of mathematical ability. The psychology of algebraic and geometrical ability has, however, received some attention in America, and "Standard Tests" are available in algebra and geometry as in every other school subject. Americans have displayed the energy, have made the mistakes, but deserve the credit, of pioneers in the statistical study of educational ability. But they have not given what English teachers want. Their Standard Tests appear often directed towards the measurement of relatively unimportant capacities. Their inquiries are often narrowly conceived, and sometimes give the impression of empiricism run riot. There are, in fact, several reasons why extensive inquiries should be undertaken in this country.

Research, good or bad, is well cared for by American educationists. Nearly every large University Training Department has its Bureau of Educational Research, where statistical inquiries can be organised on a large scale. May we add to the plea for statistical work, which appeared in the *Mathematical Gazette* a short time back, the suggestion that the Association form a Committee of Research which would provide the machinery for such work? There are many problems of psychological interest and of practical importance to teachers of mathematics that can most readily be solved statistically, problems that hardly fall within the province of the Teaching Committees, but which nevertheless require the corporate action of a body such as the Mathematical Association for adequate solution.

3. To the making of riddles there is no end, yet we will, to clear the ground, hazard a few questions that are certainly not all idle. It is not suggested that all of the following questions can be solved statistically; some of them are unlikely to be solved at all for a long time to come. It is, moreover, unnecessary in this country (in America the case might be different!) to point out that statistical methods are not a Royal Road to the truth in all matters. Too many statistics often mean too little thought, but no weapon of research should be discarded until it has served its purpose.

(*a*) To what extent are there special abilities corresponding to the recognised divisions of mathematics—an arithmetical, an algebraic, a geometrical ability, and so forth ?

(*b*) Or, what broad types of mathematical ability, possibly cutting across the divisions between types of subject matter, are there ?

(*c*) Is there a general mathematical ability ?

(*d*) What are the relations between the abilities exercised in the various branches of mathematics, and between these abilities and non-mathematical abilities ?

(*e*) How shall we test mathematical ability apart from mathematical knowledge, supposing that " ability " has meaning apart from knowledge ?

(*f*) How far can mathematical ability be trained, and to what extent is it fixed at birth ?

(*g*) Is it possible, and if so, is it desirable, to frame Standard Tests to measure all kinds of mathematical ability ?

(*h*) What are the relations between mathematical abilities and " Intelligence " ?

(*i*) How far do the assessments of mathematical ability obtained by means of ordinary examinations (including both school examinations and " external " public examinations) agree with the results obtained with tests designed and calibrated by statistical methods ?

4. The writer hopes that readers will take up the discussion of some or all of these topics in the *Mathematical Gazette*, and adds the following brief notes as further matter for criticism. The notes (*a*), (*b*), etc., refer to the questions (*a*), (*b*), etc., respectively in § 3.

(*a*) and (*b*) In so far as effective ability is dependent upon knowledge, it is obvious that the results of a test in, for example, analytical geometry will bear a close relation to results in both algebraic and geometrical tests. Dr. Brown, however, finds a correlation of $-\cdot05$ between algebraic and geometrical ability, and only $\cdot23$ between geometry and arithmetic. It is very probable that if the complex abilities called " algebraic " and " geometric " could be resolved into parts A, B, C, ..., and a, b, c, ... respectively, we should find higher correlations than Dr. Brown's between some of the parts ; that the correlation between, for instance, the algebraic B and the geometric c was higher than that between $A+B+C+\ldots$ and $a+b+c+\ldots$, which is what Dr. Brown evaluates.

(*c*) The only statistical criterion for the existence of a general factor, such as general mathematical ability, hitherto proposed, is that of Spearman, which has been used by Burt in his researches into educational abilities ; the validity of that criterion has been questioned by Prof. Godfrey Thomson. In any case we can hope to discover a general mathematical ability only by resolving mathematical thinking into relatively specific processes, and then working out the correlations between the results of tests designed to measure those processes.

(*d*) A related problem is that of the " transfer of training " between different sections of mathematics and between mathematics and other subjects. Where there is subject matter in common to two studies—as between Addition and Subtraction (the common element of " carrying," for example)—training in the one will probably improve efficiency in the other, though it has been shown by Sleight and others that this is not necessarily the case.

(*e*) and (*f*) These questions are of academic rather than immediate practical interest. What is the innate basis of mathematical ability ? To what extent, if at all, do we inherit our numerical and spatial ways of thinking ? Problems of this kind have long enough been the monopoly of philosophers, but psychologists have as yet no satisfactory answers to offer.

(*g*) and (*i*) The writer believes that Standard Tests serve a very useful purpose, though they cannot replace class tests set by the teacher. With

regard to such tests observe: (i) The Norms for American tests are expressed in terms of "Grades," and are therefore inapplicable to English children. We require norms obtained with English children and expressed on an age basis. (ii) Whether tests are used for comparison of schools or methods of teaching, or for diagnosis of pupils' difficulties, or for research purposes, a number of "comparable" tests (tests of equal difficulty and dealing with the same kind of subject matter) should be applied. The framing of a considerable number of comparable tests and their calibration is a work that the Mathematical Association is especially well fitted to perform. It would first be necessary to discover whether there was any consensus of opinion as to what constituted a good test, and whether it is possible to devise tests applicable equally well to all schools. If this is not possible there would seem to be no justification for the labour of designing standard tests.

(*h*) In every Intelligence Test, whether an Individual or Group Test, we find mathematical problems. The whole subject of Intelligence testing is badly in need of critical analysis, and such analysis would probably result in the exclusion of many problems of a mathematical nature that are at present found in Intelligence tests. One example of what seems a *legitimate* test of a mathematical kind may be taken from the Stanford Revision of Binet's tests. A visiting card is shown to the testee, and also two triangles obtained by cutting a similar card diagonally. The child has to place the two triangles together to form a figure like the whole card. It appears that the average child of five years can do this, but the average child of any less age cannot. The possibility of such definite age assignment of a test is a necessary, but not a sufficient, condition for the test to be an Intelligence test. The visiting card test is acceptable as an Intelligence test, because the age assignment is practically independent of the schooling the child has received, because the answering of the question is an act that we should in commonsense regard as intelligent, and because the mental processes involved in the solution of the test (comparison of forms, trial of "different combinations under the influence of the directing idea," and the keeping "in mind the end to be attained, that is to say, the figure to be formed") are of a kind found in a very large number of cases of intelligent action. On the other hand, the following can scarcely rank as Intelligence tests at all: "If a man runs 100 yards in 10 seconds, how many feet does he run in $\frac{1}{5}$ of a second?" (Terman Group Test); and: "A hotel serves a mixture of 3 parts cream and 2 parts milk. How many pints of cream will it take to make 25 pints of the mixture?" (Otis Group Intelligence Scale). The possibility of correctly answering these two questions clearly depends very much upon whether the child has been well or badly taught in arithmetic.

5. Their zeal for mental measurement and for statistical methods has no doubt carried some American inquirers to untenable positions, from which a broader view of the subjects investigated would have saved them. The value of statistical study in the investigation of so difficult a problem as the nature of mathematical ability is, however, unquestionable, and the formation of a Committee of Research, composed of teachers who are prepared to collect and analyse data, is essential before much can be achieved by statistical methods. Whether it is desirable to frame Standard Tests in mathematics applicable in this country is a matter that such a Committee could decide. Into the question of the relations between mathematics and "Intelligence" the writer is carrying out an inquiry in which he would be grateful for the assistance of any readers of the *Mathematical Gazette* who are interested.

University College of North Wales, Bangor. E. R. HAMILTON.

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185. Gaisford of *Christ Church* to an undergraduate who has told him what he proposed to read: "Oh, mathematics!—well, they will keep you out of mischief."

CALCULUS STENCILS.

By Prof. G. H. Bryan, F.R.S.

I.

1. $\frac{dy}{dx}$ = limit of $\frac{y_2-y_1}{x_2-x_1}$ or $f'(x)$ = limit of $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ when (x_1, y_1) and (x_2, y_2) approach limit (x, y), if limit in question unique. Thus limits of $\frac{f(x+h)-f(x)}{h}$, $\frac{f(x)-f(x-h)}{h}$ and $\frac{f(x+h)-f(x-h)}{2h}$, when $h\to 0$, must all become $=f'(x)$ or $\frac{df(x)}{dx}$.

2. Ex. Let $y=A+Bx+Cx^2+Dx^3+Ex^4$ etc.;

$$\therefore \frac{y_2-y_1}{x_2-x_1}=B+C(x_2+x_1)+D(x_2^2+x_2x_1+x_1^2) \text{ etc.},$$

and $$dy/dx=B+2Cx+3Dx^2+4Ex^3 \text{ etc.}$$

3. Function of Function. $\frac{y_2-y_1}{x_2-x_1}=\frac{y_2-y_1}{u_2-u_1}\times\frac{u_2-u_1}{x_2-x_1}$, giving

$$dy/dx=dy/du\times du/dx.$$

4. Product Rule. $y=uv$. (See Fig. 1.)

$$\frac{y_2-y_1}{x_2-x_1}=\frac{u_2v_2-u_1v_1}{x_2-x_1}=\frac{u_2v_2-u_1v_2+u_1v_2-u_1v_1}{x_2-x_1}=u_2\frac{u_2-u_1}{x_2-x_1}+u_1\frac{v_2-v_1}{x_2-x_1},$$

giving $dy/dx=v\,du/dx+u\,dv/dx$ in limit.

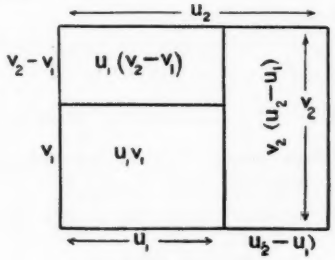

Fig. 1.

5. Quotient. $y=\frac{u}{v}$; $\therefore\ u=vy$.

$$\frac{du}{dx}=v\frac{dy}{dx}+y\frac{dv}{dx};\quad \therefore\ \frac{dy}{dx}=\left(\frac{du}{dx}-y\frac{dv}{dx}\right)\Big/v,$$

or $$\frac{dy}{dx}=\frac{v\,du/dx-u\,dv/dx}{v^2}.$$

Alternative proof. $\frac{y_2-y_1}{x_2-x_1}=\frac{\frac{u_2}{v_2}-\frac{u_1}{v_1}}{x_2-x_1}=\frac{u_2v_1-u_1v_2}{v_2v_1(x_2-x_1)}$

$$=\frac{u_2v_1-u_1v_1+u_1v_1-u_1v_2}{v_2v_1(x_1-x_1)}=\frac{v_1(u_2-u_1)}{v_2v_1(x_2-x_1)}-\frac{u_1(v_2-v_1)}{v_2v_1(x_2-x_1)}, \text{ etc.}$$

6. POWER.

$$\frac{dx^2}{dx}=x\frac{dx}{dx}+x\frac{dx}{dx}=2x,\quad \frac{dx^3}{dx}=\frac{d(xx^2)}{dx}=x\frac{dx^2}{dx}+x^2\frac{dx}{dx}=2x^2+x^2=3x^2,\ \text{etc.},$$

and if $\frac{dx^n}{dx}=nx^{n-1}$,

$$\frac{dx^{n+1}}{dx}=\frac{d}{dx}(x^n x)=x\frac{dx^n}{dx}+x^n\frac{dx}{dx}=xnx^{n-1}+x^n=(n+1)x^n,$$

as in mathematical induction.

7. ROOT. $y=x^{p/q}=\sqrt[q]{x^p}$ when $y^q=x^p$; $\therefore\ qy^{q-1}\frac{dy}{dx}=px^{p-1}$;

whence $\frac{dy}{dx}=\frac{p}{q}\frac{x^{p-1}}{y^{q-1}}$ reduces to $\frac{p}{q}x^{p/q-1}$ or nx^{n-1}, where $n=\frac{p}{q}$.

8. NEGATIVE INDEX. $y=x^{-n}=1/x^n$; $\therefore\ yx^n=1$; $\therefore\ x^n\frac{dy}{dx}+ynx^{n-1}=0$.*

$$\frac{dy}{dx}=-\frac{nux^{n-1}}{x^n}\ \text{reduces to}\ -nx^{-n-1}.$$

CARTESIAN EQUATIONS OF TANGENTS.

II.

1. CHORD. $\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$ gives in limit Tangent $\frac{y-y_1}{x-x_1}=\frac{dy}{dx}$ at (x_1y_1). Briggs & Bryan, *R.L and* ⊙.

[*P*] $y^2=4ax$, $2y\frac{dy}{dx}=4a$,

$$y-y_1=\frac{2a}{y_1}(x-x_1),$$

$$yy_1=2ax+y_1^2-2ax_1$$
$$=2ax+4ax_1-2ax_1$$
$$=2a(x+x_1),$$

[*E*] $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$,

$$y-y_1=-\frac{b^2x}{a^2y}(x-x_1)$$

$$\frac{(x-x_1)x_1}{a^2}+\frac{(y-y_1)y_1}{b^2}=0,$$

$$\therefore\ \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1.$$

[*H*] write $-\beta^2$ for b^2;

Ex. Curve $xy=c^2$; prove tangent $xy_1+yx_1=2c^2$.
Gen. Eq. 2 deg.

$$ax^2+2hxy\qquad+by^2\qquad+2gx+2fy\quad+c=0,$$

$$2ax+2h\left(x\frac{dy}{dx}+y\right)+2by\frac{dy}{dx}+2g\quad+2f\frac{dy}{dx}+0=0.$$

$\frac{dy}{dx}=-\frac{ax+hy+g}{hx+by+f}$ Tangent reduces to

$$axx_1+h(xy_1+yx_1)+byy_1+g(x+x_1)+f(y+y_1)+c=0,$$

making use of fact that x_1y_1 is on curve.

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* Dif. co. of constant = 0.

2. SUBTANGENT (*TN*), SUBNORMAL (*NG*).

[*P*] If $y=0$, $x+x_1=0$;

$x=-x_1$ or $AT=-AN$, § 32,

On axis $x=0$, [*P*] $At=\frac{1}{2}AM$ or An

[*E, H*] $y=0$, $\frac{xx_1}{a^2}=1$, $xx_1=a^2$,

or $CN \cdot CT=CA^2$, § 33.

[*E, H*] $Cn \cdot Ct=b^2$ or $-\beta^2$.

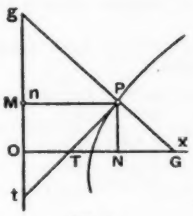

FIG. 2.

Equation of normal. Brigg and Bryan form $B(x-x_1)-A(y-y_1)=0$.

[*P*] $2a(y-y_1)+y_1(x-x_1)=0$;

Put $y=0$, $x-x_1=2a$, § 34.

[*E, H*] $\frac{y_1}{b^2}(x-x_1)-\frac{x_1}{a^2}(y-y_1)=0$.

$y=0$, $x-x_1=-\frac{b^2}{a^2}x_1$, $x=e^2x_1$, § 35.

In calculus, equation of normal is $(y-y_1)\frac{dy_1}{dx_1}+x-x_1=0$, giving subnormal

$$(x-x_1)_{y=0}=y_1dy_1/dx_1,$$

Exs. (i) $xy=c^2$, $OT=2ON$, $Ot=2On$, $\Delta TOt=2c^2$;

(ii) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, $Tt=a$; (iii) $\sqrt{x}+\sqrt{y}=\sqrt{a}$, $OT+Ot=a$;

(iv) $y=kx^m$, $TN=\frac{ON}{m}$, $tn=m \cdot On$.

"TANGENT" AND "PERPENDICULAR" FORMS (*Conics only*).

"$y=mx+c$," "$B^2-4AC=0$."

[*P*] $(mx+c)^2=4ax$, $(2mc-4a)^2=4m^2c^2$, $2mc-4a=-2mc$, $c=a/m$;

$$\therefore\ y=mx+\frac{a}{m}.$$

[*E*] $\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$;

$\therefore \left(\frac{2mc}{b^2}\right)^2=4\left(\frac{1}{a^2}+\frac{m^2}{b^2}\right)\left(\frac{c^2}{b^2}-1\right)$ gives $c^2=a^2m^2+b^2$, $y=mx\pm\sqrt{(a^2m^2+b^2)}$.

Put $m=-\cot\alpha$,

$x\cos\alpha+y\sin\alpha=\pm\sqrt{(a^2\cos^2\alpha+b^2\sin^2\alpha)}$, $px+qy=\pm\sqrt{(a^2p^2+b^2q^2)}$ also.

[*H*] $y=mx\pm\sqrt{(a^2m^2-\beta^2)}$, real if $|m|>\beta/a$. If $m=\pm\frac{\beta}{a}$ equation is $y=mx$.
Put $y=mx+c$, $\left(\frac{1}{a^2}-\frac{m^2}{\beta^2}\right)x^2-\frac{2mc}{\beta^2}x-\frac{c^2}{\beta^2}-1=0$.

[ASYMPTOTE] (§§ 53-55) with $c=0$, $m^2=\beta^2/a^2$ gives $0.x^2+0.x-1=0$. Both roots $=\infty$. (See *Tutorial Algebra.*)

III.

INTEGRAL. If $dy/dx = u$, then $y = \int u\,dx$, logically $y = \dfrac{d^{-1}u}{dx^{-1}}$.

Const. of Integration. $\dfrac{d}{dx}(y+c) = \dfrac{dy}{dx} + 0 = \dfrac{dy}{dx}$;

$\therefore$ $y+c$, another value of $\int u\,dx$.

Geometry. $y = f(x)$ and $y = f(x) + c$ have same slope. Any value of c determines a *Particular Integral.*

Ex. If $y = x^n$, then $dy/dx = nx^{n-1}$; but if $dy/dx = nx^{n-1}$, y may be x^n or $x^n + c$; $\therefore$ $dy/dx = x^n$ when $y = \dfrac{n^{n+1}}{n+1} + c$, and *particular integral* is

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}.$$

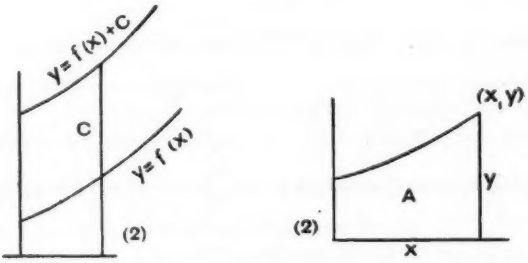

FIG. 3

AREAS. A as Fig. 3 above; then in Figs. 4, 5, $A_2 - A_1$ lies between rects. PN and QM, *i.e.* between $(x_2 - x_1)y_1$ and $(x_2 - x_1)y_2$; $\therefore$ $\dfrac{A_2 - A_1}{x_2 - x_1}$ between y_1 and y_2; $\therefore$ in limit $\dfrac{dA}{dx} = y$; $\therefore$ $A = \int ya'x + c$, where c depends on point *from* which area is measured up to (x, y).

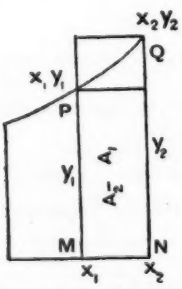

FIG. 4.

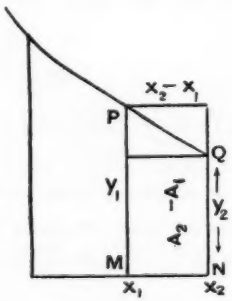

FIG. 5.

DEFINITE INTEGRAL. Let $\varphi(x)+c$ be particular integral of $f(x)$ corresponding to area of $y=f(x)$ measured from Ii; then area $iA=\varphi(a)+c$, area $iB=\varphi(b)+c$; $\therefore$ area between $x=a$ and $x=b$ is $\varphi(b)-\varphi(a)$, written

$$\int_a^b y\,dx \quad \text{or} \quad \int_a^b f(x)\,dx.$$

For y, $\int_a^b x^n dx=\frac{b^{n+1}-a^{n+1}}{n+1}$.

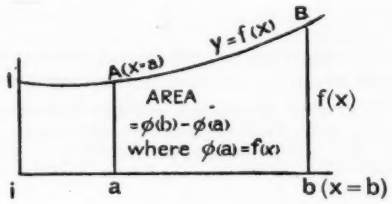

FIG. 6.

AREA INTEGRAL AS LIMIT OF A SUM OF PRODUCTS. (Figs. 7, 8.)

rect. $P_0M_1=y_0\Delta x_1$, $P_1M_2=y_1\Delta x_2$,

rect. $M_0P_1=y_1\Delta x_1$, $M_1P_2=y_2\Delta x_2$.

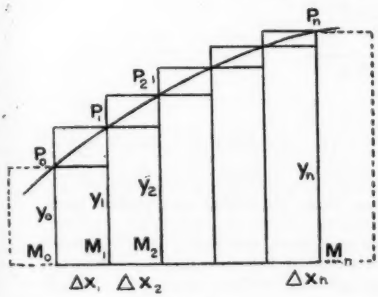

FIG. 7.

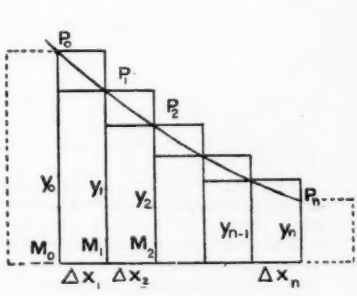

FIG. 8.

(i) $y_0\Delta x_1+y_1\Delta x_2+\ldots y_{n-1}\Delta x_n <$ curve area Fig. 7, $>$ area Fig. 8.

(ii) $y_1\Delta x_1+y_2\Delta x_2+\ldots y_n\Delta x_n >$ curve area Fig. 7, $<$ area Fig. 8.

Form curve area between the two. Let Δx_1, $\Delta x_2 \ldots$ be made each $<\epsilon$, or y_1-y_0, y_2-y_1, etc., each $<\eta$; then difference of the two sums (i), (ii) $<\epsilon(y_1-y_0+y_2-y_1\ldots+y_n-y_{n-1})$, *i.e.* (y_n-y_0), and is also

$$<\eta(\Delta x_1+\Delta x_2+\ldots+\Delta x_n), \quad \textit{i.e.}\ \eta\cdot M_0M_n.$$

Both ϵ and η can diminish indefinitely by making divisions smaller, and both products (i) (ii) become equal in limit, and therefore equal curve area $\int y\,dx$.

(*Note.*—The student should notice the effect of shifting the rectangles of Figs. 7, 8 one space to the right or left. One rectangle represented by dotted lines will come into the limits of integration, and one will go outside

into the space shown on the other side by the dotted lines. Also the intermediate rectangles if less than the curvilinear area will now occupy spaces greater than that area, or if greater, then less.

It follows that the difference between the rectangle-sums which are greater and less than the curvilinear area is less than the difference of the rectangles added and taken off when these parts of the figure are shifted through one division. This difference can therefore be made as small as we like by making the sections sufficiently thin.)

IV.

If $dy/dx = f'(x)$, can we say $dy = f'(x)\,dx$, *e.g.* for area of curve $dA = y\,dx$?

Def. of NEARLY EQUAL ($\doteqdot$) and LIMITING EQUALITY ($a \to b$):

Def. (i). $X \doteqdot A$ when $X \sim A$ is a small fraction of A or of X. Thus 92,000,000 miles $\doteqdot$ 92,001,000 miles, since difference $= \frac{1}{92000}$ of first; but 0·0002 inch is not $\doteqdot$ 0·0003 inch, since difference is only $\frac{1}{2}$ of first, $\frac{1}{2}$ not being very small.

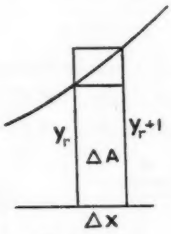

FIG. 9.

Put $X \sim A = A\epsilon$, thus $\frac{X}{A} \sim 1 = \epsilon$ or $\frac{X}{A} = 1 \pm \epsilon$; thus when $X \doteqdot A$, ϵ is a small fraction of unity.

Def. (ii). If by any process ϵ can be made less and less without stopping at any value however small, short of zero, then X, A become equal in limit, or $X \to A$.

This definition applies when the same process makes both X and A vanish or become infinite. Thus $2x + x^2 \to 2x$ when $x \to 0$, for $(X - A)/[A = x^2/2x = x/2$ becomes 0 when $x = 0$, but $2x$ and $3x$ are not equal in limit when $x \to 0$, though both become $= 0$.

In definition of dy/dx, put $x_2 - x_1 = \Delta x$, $y_2 - y_1 = \Delta y$; then condition that $\Delta y \to f'(x)\,\Delta x$ requires $\frac{\Delta y}{f'(x)\,\Delta x} - 1$ should vanish in limit, viz. $\frac{\Delta y}{f'(x)\,\Delta x} \to 1$, or $f'(x)$ is limit of $\Delta y/\Delta x$, which is same as $\frac{dy}{dx}$, where dx and dy stand for limits of Δx and Δy when both become infinitesimal.

For curvilinear area (figure above) ΔA is between rect. $y_r \Delta x$ and rect. $y_{r+1}\Delta x$, and condition of limiting equality makes $\epsilon = \frac{y_{r+1} - y_r}{y_0}$, which may be made as small as we please by bringing the points closer if we exclude element from points where y changes suddenly. With this condition we say $dA = y\,dx$.

Examples. If velocity of point is v, *we say* that in time dt it goes a distance $v\,dt$; or, if the acceleration is f, that change of velocity in time dt is given by $dv = f\,dt$.

V.

1. LIMIT OF A SUM. If $X_1 \to A_1$, $X_2 \to A_2$, etc., then $\Sigma X \to \Sigma A$.

Proof. Let $X_1 - A_1 = A_1\epsilon_1$ (*not* ϵ, as in Osgood), $X_2 - A_2 = A_2\epsilon_2$, etc., and let η be the numerically greatest value of ϵ_1, ϵ_2, etc.; then

$$X_1 - A_1 < A_1\eta, \text{ etc., and } \Sigma X - \Sigma A < \eta\Sigma A_1; \quad \therefore \frac{\Sigma X - \Sigma A}{\Sigma A} < \eta,$$

where η approaches limit 0, limit of $\Sigma X \to \Sigma A$.

2. ORDER OF INFINITESIMALS. X infinitesimal relative to A if in limit $X/A \to 0$. Put $X = \alpha A$, and let Y be another infinitesimal with $Y/A = \beta$. Then if $\lim \beta/\alpha^n$ is finite, β is of the n^{th} order relative to α, and Y of n^{th} order relative to X and A.

3. TRIANGLES WITH INFINITESIMAL ANGLES.

(i) *Right angled.* $CB^2 = AB^2 - AC^2 = (AB - AC)(AB + AC)$;
$\therefore AB - AC = CB^2/(AB + AC)$; $\therefore$ $AB - AC$ is of second order in CB.

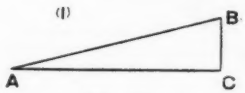

FIG. 10.

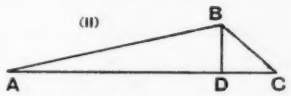

FIG. 11.

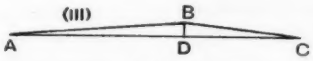

FIG. 12.

(ii) *C finite.* $DC = BD$ cot C of same order as BD, but $CD = AC - AD$ differs only in second order from $AC - AB$, or to first order $AC - AB = DC$.

(iii) *A, C both infinitesimal.* $AB - AD$ and $BC - DC$, both of second order; $\therefore AB + BC$ differs from AC in second order only relative to BD.

4. LENGTH OF A CURVE, defined as limit of perimeter of inscribed or circumscribed rectilineal figure with infinite number of infinitesimal sides. In fig. $AB < AC + CB < AD + DC + CB$, an inscribed perimeter increases with number of sides; and since $QP + PR > QR$,

$$AP + PB > AQ + QR + RB > AS + ST + TR + RB;$$

$\therefore$ sum of tangents decreases with number of sides.

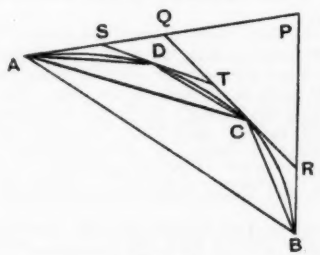

FIG. 13.

But the triangles approach limiting forms of § 3 (iii); ∴ by § 1 above, sums of tangents and of chords are ultima, *i.e.* equal, and each represents length of curve.

5. By Stencil 4, element of length ds is limit of chord PP', and in Cartesians $ds^2 = dx^2 + dy^2$, $\sin\varphi = dy/ds$, $\cos\varphi = dx/ds$, etc. (see fig. 14).

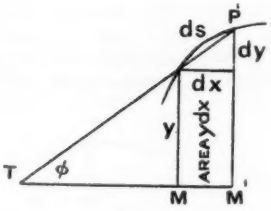

FIG. 14.

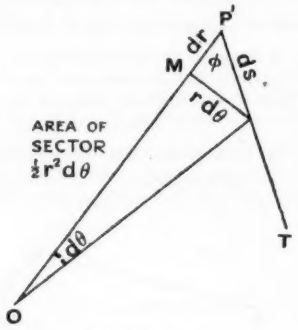

FIG. 15.

Also in polars $PM = r\,d\theta$ (radians),

$$P'M = dr \text{ (§ 3, ii)}, \quad \sin\varphi = r\,d\theta/ds,$$

$$\cos\varphi = dr/ds, \quad ds^2 = dr^2 + r^2 d\theta^2, \quad \text{area of sector} = \tfrac{1}{2}r^2 d\theta.$$

Lastly, $$s = \int \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = \int \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta, \text{ etc.}$$

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186. I was the worst caligrapher and the most blundering arithmetician. Indeed, I had never learnt arithmetic, beyond a weekly lesson which I had always contrived to shirk. . . . I soon, however, conquered the mysteries of figures [æt. 15.]—E. B. Lytton (*Life*, by his grandson, 1911, i. 48).

187. Tom Tulliver was only thirteen, and had no decided views in grammar and arithmetic, regarding them for the most part as open questions. . . . It was (Mr. Stelling's) favourite metaphor that the classics and geometry constituted that culture of the mind which prepared it for the reception of any subsequent crop. . . .

Tom, more miserable than usual, determined to try his sole resource. . . . He paused a little to consider how he should pray about Euclid—whether he should ask to see what it meant, or whether there was any other mental state which would be more applicable to the case. But at last he added: "And make Mr. Stelling say I shan't do Euclid any more. Amen." . . . "I don't think I *am* well, father," said Tom, "I wish you'd ask Mr. Stelling not to let me do Euclid—it brings on the toothache, I think." "Euclid, my lad—why—what's that?" "O, I don't know; it's definitions, and axioms, and triangles, and things. It's a book I've got to learn in—there's no sense in it."

Tom gave up praying that he might learn one part of his work: "Please make Mr. Stelling say I am not to do Mathematics."—G. Eliot, *The Mill on the Floss*.

MATHEMATICAL NOTES.

679. [I. a.] If p is small, $10^p - 1$ is approximately equal to $p \log_e 10$: hence if $\pm p$ is the possible error in a common logarithm L, the consequent part of the possible error in its antilogarithm is $\pm 10^L p \log_e 10$.

If the amount at $r\%$ compound interest for n years be calculated from the formula $A = P\left(1 + \frac{r}{100}\right)^n$ by N-figure tables, the possible error in the sum of the logarithms of the right-hand side is $\pm \frac{1}{2}(n+1)10^{-N}$; and the consequent part of the possible error in the antilogarithm is

$$\pm P\left(1 + \frac{r}{100}\right)^n \cdot \frac{n+1}{2} \cdot 10^{-N} \cdot \log_e 10.$$

The proper part of its possible error is $\pm \frac{1}{2}\, 10^{k+1-N}$, where k is the characteristic of $\log_{10} A$.

Thus the total possible error q in the result is

$$q = \tfrac{1}{2}\, 10^{-N}\left\{P\left(1 + \frac{r}{100}\right)^n (n+1)\log_e 10 + 10^{k+1}\right\}.$$

It is assumed that the tables can be relied on to give any logarithm or antilogarithm correct to N decimal places.

The above formula is a practical example of a discontinuous function: thus when $N=4$, $r=3$, $n=15$ and P passes through the value 0·6418625, q jumps from 0·0018921 to 0·0023421.

The following numerical results give indications of the limits of reliability of logarithmic tables in calculating compound interest.

I. Maximum values of n			$(r=3)$	$(r=6)$
$N=4$	$q=\frac{1}{2}$d.	$P=$£1	9	7
	$q=10$/-	$P=$£1	120	70
		$P=$£100	20	15
$N=7$	$q=\frac{1}{2}$d.	$P=$£1	159	90
		$P=$£100	45	29
	$q=10$/-	$P=$£1	321	173
		$P=$£10,000	63	39

II. Maximum value of P			$(r=3)$	$(r=6)$
$N=4$	$q=10$/-	$u=10$	£264	£198
$N=7$	$q=\frac{1}{2}$d.	$u=10$	£930	£698
	$q=10$/-	$u=10$	£264,400	£198,415

III. Minimum values of N			$(q=\frac{1}{2}$d.)	$(q=10$/-)
$r=6$	$u=25$	$P=$£1	5	3
		$P=$£1000	8	6
		$P=$£1,000,000	11	9

Haileybury College. C. W. ADAMS.

680. [V.] *Mathematical Symbols* (*v. Math. Gaz.* xi. p. 315). *Symbols for "Therefore" and "Greater" or "Less than."*

In partial answer to Query No. 655 I desire to point out that both forms ∵ and ∴ are found in Thomas Brancker's translation, brought out in London in 1668, under the title, *An Introduction to Algebra, translated out of the High-Dutch* [of Rhonius]* *into English*, much altered and amended by D[r. John] P[ell]. On p. 37 one reads: "Which Deduction is hereafter noted with [∵] that is as much as [*ergo*] only [∵] is set in the work, [*ergo*] in the Margin." One finds ∵ used in many parts of the book, but on pages 153-155 there occurs forty-one times the sign ∴ having the same signification as the ∵. Rahn's

* J. H. Rahn.

original edition, the *Teutsche Algebra*, Zurich, 1659, contained the three dots, but I do not now have access to it. The three dots are therefore of Swiss origin, and, like the symbol ÷ for division, reached England through the translation of Rahn's book. The ∵ as signifying "therefore" is given in Oughtred's *Opuscula mathematica hactenus inedita*, Oxford, 1677, p. 151; in John Alexander's *Synopsis algebraica*, London, 1693, p. 7; in John Ward's *Young Mathematician's Guide*, 1707, p. 189. Both forms ∵ and ∴ stand for "therefore" in William Jones' *Synopsis*, 1706, pp. 179, 186; and in Robert Woodhouse's *Principles of Analytical Calculation*, Cambridge, 1803. During the eighteenth century ∵ was used for "therefore" about as often as was ∴. I have not yet been able to find an eighteenth century ∵ which signified "because" or "since." This usage seems to have been introduced in Great Britain in the nineteenth century.

The ⊏ for >, and ⊐ for <, first used by W. Oughtred in his *Key* or *Clavis mathematicae* of 1647, have received so many different forms in later works, that confusion has arisen as to the significance of some of the forms used. The first deviation from Oughtred's original forms is in "Fig. EE" in the appendix, called the *Horologio*, to Oughtred's *Clavis*, where in the edition of 1647 there stands ⊏ for <, and in the 1652 and 1657 editions there stands ⊏ for <. In the text of the *Horologio*, in all three editions, Oughtred's regular notation is adhered to. John Wallis, in his *Algebra* (1685, p. 127), gives ⊐ for >, ⊐ for <. In Isaac Barrow's *Euclid*, ⊏ means >, ⊐ means <. Barrow's signs appeared in the algebras of John Kersey (1674) and Richard Sault (circ. 1694). E. Cocker in his *Artificial Arithmetick* (1684, p. 278) writes ⊏ and ⊐ for > and <, respectively. An adherence to Oughtred's original forms is found in the arithmetic of Samuel Jeake (1696). In Newton's *De analysi per aequationes*, as printed in Collins' *Commercium epistolicum* (1712, p. 20), one finds $x \sqsupset \frac{1}{2}$, which is interpreted as meaning $x > \frac{1}{2}$ in Newton's *Opuscula* (Castillion's ed., 1744, Vol. I., p. 27), and in Biot and Lefort's 1856 edition of the *Commercium epistolicum*, p. 74. But G. Eneström in *Bibliotheca mathematica*, 3. S., Vol. 12, p. 74, argues that Newton followed his teacher Barrow in the use of ⊐, and actually took $x < \frac{1}{2}$, as is demanded by the reasoning.

University of California. FLORIAN CAJORI.

681. [A. 5.] *Note* 675. *An Elucidation.*

For the sake of being intelligible, may I explain that the forward reference in Note 630 (p. 88 of this volume) was to Note 675 (p. 342)? Expecting that the latter would be set up before the former was published, I wrote "N/Z^2 in Note ... below," but the opportunity to fill the blank did not occur, and the printer just closed the gap. E. H. NEVILLE.

Aug. 1, 1923.

682. [K¹. 21. a. α.] *Note* 673, p. 338.

This is not correct as a *general* solution with the second line as it stands. Instead of "take C as near the middle as possible to avoid fractions" it should be "*take C $\frac{1}{2}$ unit from middle point.*" If $AO = x$, $AC = y$, where C is $\frac{1}{2}$ unit from middle point $x - y = 1$,

$$\therefore\ x^2 - y^2 = x + y \quad \text{or} \quad CO = \sqrt{AB \times 1}.$$

J. M. CHILD.

683. [V. 1. a.] *An Attempt to make Mathematics interesting.*

As is suggested by the writer under this heading in *Gazette*, No. 165, Note 677, there are too few books on Mathematics, especially among those written for schools, to which we can turn for reading. We have had mathematical books in dialogue, in verse and in cut-and-dried rules, but writers have not in general presented this subject in a literary form to show it in its history, romance or beauty, although it has grown out of a study of the universe and its motion.

the practical needs of man and the abstract thought of the leisured. A teacher in the class-room more frequently makes the subject live through her own appreciation of its interest, or from pity for bored and over-"ruled" children.

The "Mathematics Club" with its disconnected programme and matter, not always on a high plane, can do much to foster interest in the subject. The following is the programme of a Mathematics Club for Forms below the Sixth, which was tried as an experiment by the writer. A serious fault is that the work falls on the teacher, and suggestions to make the pupils' part more active would be welcomed.

1. Notations—pictorial and others.
2. Lantern slides; Portraits of Thales, Euclid, Pythagoras, Archimedes, Newton, Pascal, Descartes (lent privately) with a few words on the life and work of each.
3. Mathematical Rhymes and Jingles—Nursery counting rhymes, counting out games, accumulatives, popular rhymes and distiches, arithmetical rules and problems given in verse—collected from books of rhymes and arithmetics from the sixteenth century onwards.
4. The history of our weights and measures.
5. Puzzles—from Dudeney, *Number Stories of Long Ago* (D. E. Smith), and periodicals.
6. Shakespeare's Mathematics.
7. To make a simple calculating machine.
8. Exhibition of mathematical books—photographs of MSS. and the principal books from the *Grounde of Artes* to the present time.
9. Mathematics, frivolous and otherwise, in modern literature.

F. A. YELDHAM.

684. [V. 1. a. 3.] "*The Complete Angle.*" (Note 656, p. 275, *Math. Gazette*, March 1923.)

It is not easy to enter into discussion at this distance; but I would like to say something in reply to Professor Neville's Note:

(i) The statement that this "conception of an angle . . . is familiar in advanced work" may convey an impression not quite in accordance with the facts. The conception is, as I said [Note to § 12 (ii); *Gazette*, p. 192] implicit in Projective Geometry; but it is hardly more explicit to-day than in Townsend's *Modern Geometry*, where I first came upon it some twenty years ago.*

(ii) I agree that the term "line-pair" is appropriate—corresponding to the notation (l_1, l_2); but for the elementary Euclidean geometry with which I was concerned I found that the notation *AOB*, and the corresponding term "Complete Angle," made the proper transition from the familiar elementary theory; it legitimately retains the term "Angle" in standard angle-theory.† The principle of the terminology is this: (1) Complete Line; (2) Complete Angle, or Di-lateral, or Line-pair; (3) Complete Triangle, or Tri-lateral (to which attaches all the "Modern" geometry of The Triangle); (4) Complete Quadrilateral. As to (4), "Complete Quadrilateral" surely means *complete four-line figure*—not merely "complete" as to "vertices."

(iii) As to "cross," I should be in some doubt. The term "cross" is already firmly established in Geometry, in "cross-ratio" and "cross-section." It would seem more appropriate for the *point*, for which the less appropriate term "meet" is sometimes used.

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* The extreme condensation of my work for the *Gazette* necessitated the omission of a good deal of explanatory matter. Some consideration of the literature on the subject is given in a paper which will, I hope, soon be published.

† The bisector of the Simple Angle *AOB*, and the two bisectors of the Complete Angle—both referring to Angle-figures, not angle-quantities—are important examples of the use of the terminology.

(iv) The proof given by Professor Neville for the Circle-theorem is clearly important, in the sense indicated by him. But the simplicity is deceptive, from the "Elementary" point of view—in that the proof turns on the intrinsically difficult geometrical theory of Ratio and Proportion. The *desideratum* is a proof as directly as possible from the elementary definition of The Circle. Such a proof (due in principle to my friend and former pupil, Mr. E. J. G. Pitman, Acting-Professor of Mathematics, Christchurch, N.Z.) is given in the paper referred to in the footnote. D. K. PICKEN.

Ormond College, Melbourne.

685. (H. 1. c.) *Note on the solution of* $F(D)y=s(x)$.

A method employed, in many text-books, to solve this equation is to write $y=\dfrac{1}{F(D)}s(x)$, and expand the operator in a power series in D. It is used when, for some value of n, $s^n(x)\equiv 0$. The method is admitted to be empirical and is used with the safeguard of verifying the solution. The empirical element is, however, easily removed, as the following will show.

$F(m)$ will be assumed not to have m as a factor. The necessary modifications when m is a factor suggest themselves at once.

If
$$\frac{1}{F(m)}=\Sigma\frac{A_{a,r}}{(m-a)^r},$$
the R.H.S. being the usual expression of $1/F(m)$ as a sum of partial fractions, it is well known that a value of $\{1/F(D)\}y$ is given by

$$\Sigma\frac{A_{a,r}}{(D-a)^r}y. \qquad \text{.....(A)}$$

Consider now $\dfrac{1}{(D-a)^r}s(x)$: if y is a possible value, then

$$(D-a)^r y=s(x),$$

or
$$D^r(e^{-ax}y)=e^{-ax}s(x), \qquad \text{.....(1)}$$

or
$$e^{-ax}y=\int\int^{\cdots(r)}e^{-ax}s(x)(dx)^r.$$

It is easily seen that, using integration by parts, the r-ple integral may be written

$$(-)^r\frac{e^{-ax}}{a^r}\left\{s(x)+B_1\frac{s'(x)}{a}+\ldots+B_{n-1}\frac{s^{n-1}(x)}{a^{n-1}}\right\}, \qquad \text{.....(2)}$$

if all the derivatives of $s(x)$ after $s^{n-1}(x)$ vanish identically.

Using Leibnitz' Theorem to differentiate (2) r times, we may, in virtue of (1), equate the result to $e^{-ax}s(x)$. We then see that the B's satisfy the system of equations

$$B_k-{}^rC_1B_{k-1}+{}^rC_2B_{k-2}+\ldots\pm B_{k-r}=0 \quad [k=1, 2, \ldots, n-1],$$

i.e. that $1+B_1x+\ldots+B_{n-1}x^{n-1}$ is a recurring series whose scale of relation is $(1-x)^r$, and that therefore 1, $B_1 \ldots B_{n-1}$ are the coefficients of 1, $x \ldots x^{n-1}$ in the expansion of $(1-x)^{-r}$.

The r-ple integral is accordingly given by

$$(-)^r\frac{e^{-ax}}{a^r}\left\{1+r\cdot\frac{D}{a}+\ldots+\frac{r(r+1)\ldots(r+n-2)}{\lfloor n-1}\cdot\frac{D^{n-1}}{a^{n-1}}\right\}s(x),$$

and so a value of $\dfrac{1}{(D-a)^r}s(x)$ is given by

$$\left(-\frac{1}{a}\right)^r\left(1-\frac{D}{a}\right)^{-r}s(x), \qquad \text{.....(B)}$$

when this operator is expanded in a power series in D.

Now $$\frac{1}{F(m)} \equiv \Sigma \frac{A_{a,r}}{(m-a)^r}.$$

For sufficiently small values of m the R.H.S. may be written as the sum of a finite number of power series in m. If, by any method, the L.H.S. may be legitimately expanded in a power series, the coefficient of m^k is equal to the sum of the coefficients of m^k in the separate power series on the right. .....(C)

But results (A) and (B) taken together show that a solution of the original equation is given by

$$\Sigma A_{a,r} \frac{(-)^r}{a^r} \left(1 - \frac{D}{a}\right)^{-r} s(x),$$

when the operators are expanded in power series in D. Hence by (C), if by any method $\frac{1}{F(D)}$ is expanded directly in a power series in D, we obtain a solution of the original equation by operating on $s(x)$ with the series thus obtained.

It is interesting to note also that a solution of $(D-a)^r y = s(x)$ is given by

$$y = \left(-\frac{1}{a}\right)^r \left(1 - \frac{D}{a}\right)^{-r} s(x),$$

where the operator is expanded in a power series in D, provided the resulting series is convergent.

Univ. Coll., N. Wales, Bangor. W. L. FERRAR.

686. [V. a. λ.] *On Note* 615 (xi. p. 58): *The Notation of the Calculus.*

The comments of this Note are justified by much loose practice. Nevertheless, the strongest possible case can be made for the "classic" notation of the Calculus, as being one of the outstanding triumphs of pioneering intuition (intuition by no means always so unerring in mathematical notation and terminology).

1. First, we have the δ-notation for *increment*-infinitesimals: a type of composite symbol which says, *in two letters, e.g.* δx, that it denotes *a variable which is an "increment" of the variable x and is being used only with a view to a "limit" proposition which results from the "tending to" 0 of this increment.* This is the very perfection of notation; but its perfection is sadly marred when this same notation is used for infinitesimals which are *not* increments—as is far too commonly done—*e.g.* for the infinitesimal chord-quantity or the normal element quantity ("δn," when there is no n of which it is increment); and the notation is also in some (perhaps not very serious) conflict with that for a possible product of $\delta . x$—as to which, however, it would appear that we must soon cease to omit the multiplication sign.*

2. The second phase of the notation is equally good—in that the limit *result* is expressed by using "d" *in* the limit where "δ" occurred (in such a way as to affect the limit) *before* the limit. For example, dy/dx **means** $\mathrm{Lt}(\delta y/\delta x)$, d^2y/dx^2 **means** $\mathrm{Lt}(\delta^2 y/\delta x^2)$,† etc.; $dy = f'(x) . dx$ **means** that, if $y = f(x)$, δy differs from $f'(x) . \delta x$ by "an infinitesimal of higher order" than either; so, again, $du = f_x'(x, y, z) . dx + f_y'(x, y, z) . dy + f_z'(x, y, z) . dz$ **means** that, if $u = f(x, y, z)$, δu differs from the suggested expression in δx, δy, δz by an infinitesimal of higher order than these. (The specification of dx as "a quantity so small that its square may be neglected" is, like the cognate phrase

* Articles in *Nature* by Sir Napier Shaw, writing on the most modern of the applied sciences, are very interesting on this point; in particular, *Nature*, Nov. 4, 1920, "Symbolic Language of Science," in which he advocates expansion of notation "by proceeding from single letters to syllables . . . *if we could do away with the convention that multiplication needs no symbol of operation.*" Many strong arguments for this can be advanced.

† It does not seem to be sufficiently emphasised that this is what justifies Leibniz' notation for the higher derived functions. The Continental phrases "quotient of differences" and "differential quotient" are very little recognised in the British world: the latter so very much better than our "differential *coefficient*." "Coefficient" is overworked by us.

"infinitely small," unsound. There is no such quantity, in any absolute sense; and the point about the "d" propositions is that they are *absolutely exact*. dx is not a quantity; it is part of a symbolism most ingeniously devised for expressing Calculus limits. The definition of the "differential" dy by $y'.\delta x$—and therefore of dx by δx—is for the same reasons unsound; it is a—certainly most plausible—"dodging" of an essential Calculus idea.)

3. Again, as "d" is related to "δ," so is "$\int$" related to "Σ." $\int_a^b f(x).dx$ **means,** of course, $\mathrm{Lt}\,\Sigma f(x).\delta x$ for a summation (Σ) extending in a certain specified way over the range (a, b) of x. It is the proper notation for "the *definite* integral," and the substitution $x=\phi(\xi)$, $dx=\phi'(\xi).d\xi$ (in the sense above indicated) is appropriate to it. But, as the Note suggests, the notation is not nearly so appropriate for "the *indefinite* integral" (or "anti-derivative"); and its elementary use in that connection tends to spoil the effect of its proper use (indicated above).

4. The use of "D" for differentiation points, as suggested, to the use of a single symbol for the inverse operator. But the symbol D^{-1}, used to some extent in Differential Equations work, belongs to a type of notation for inverses which is unsatisfactory (because in conflict with a more fundamental use of -1 in the exponent position). Why not use a reversed D—viz. Q—or I (for "integration")? *E.g.* $D\sin x=\cos x$, $I\cos x=\sin x+C$.

It is clearly not sound to use $\int$—divorced from dx—for this purpose, if we are to retain essential ideas.

5. When it is necessary, or desirable, to indicate the "argument" of a differentiation, this is, of course, commonly done by means of a suffix—*e.g.*

$$D_x y = D_u y \,.\, D_x u.$$

And, in the same way, we may write $I_x y = I_u(y \,.\, D_u x)$. This type of notation is particularly useful in the ticklish business of handling partial derivatives clearly; *e.g.* if x, y, u, v are connected by one relation, a function, z, of them, specified (say) by $f(x, y, u, v)$, is expressible in terms of any three of them—say

$$z=f(x, y, u, v)=\phi(x, u, v)=\psi(x, y, v)=\chi(x, y, u), \text{ etc.},$$

and $\partial z/\partial x$ has therefore four quite distinct possible uses, viz. for the functions specified respectively by

$$f_x'(x, y, u, v), \quad \phi_x'(x, u, v), \quad \psi_x'(x, y, v), \quad \chi_x'(x, y, u).$$

The *dash*—meaning differentiation—is commonly omitted in this notation for partial derivatives. This is at best a doubtful economy, but certainly not permissible if we are to use a corresponding inverse notation. A suffix, as in y_1 and $f_1(x)$, has been to some extent used for the anti-derivative (*e.g.* in the extended theorem of "Integration by Parts"), but this clashes both with the use in $y_1=f(x_1)$—for particular values x_1, y_1 of x, y—and with the suffix notation (as above) for partial derivatives. Possibly, as the *dash* is the French "acute" accent, so the French "grave" accent might be used as its inverse—so that we should have

$$dy/dx=y'=Dy=Df(x)=f'(x),*$$

$$\int f(x)\,.\,dx=y\grave{}=Iy=If(x)=f\grave{}(x).$$

D. K. PICKEN.

687. [V. 1. a. ϵ.] *A point in teaching Quadratics.*

Students rarely ask, but none the less may wonder why, when we reach the form $x^2=a^2$, we do not write $\pm x=\pm a$. It is well to show them that as the four forms: $+x=+a$, $+x=-a$, $-x=+a$, $-x=-a$, are equivalent to the one form $x=\pm a$.

A. B. OLDFIELD.

* Each of these notations has its own use and value.

688. [X. 2.] *Direct Method for finding* $\log_{10} 3$.

3000	1	·25
9000	·5	·125
8100	·25	·0625
6480		·03125
81		·00781
6561	·125	·0005
39366		·00005
3280		·4771
394		
7		$\log_{10} 3 = ·4771$
4305	·0625	
1722		
129		
2		
1853	·03125	
1482		
93		
5		
3433	·01562	
10299		
1373		
103		
10		
1179	·00781	
118		
84		
10		
→ 139	·0039	
42		
12		
193	·0020	
174		
6		
373	·0010	
1119		
261		
11		
→139	·0005	
193	·00025	
373	·00012	
112	·00006	

To find $\log_{10} 3$ by repeated squaring (14 times).

The first column is the actual work of squaring (using approximate multiplication).

The second column gives successive powers of ·5, the underlined ones being the successive terms in the series for $\log_{10} 3$.

R. F. MUIRHEAD.

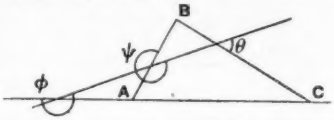
689. [M[1]. 5. c. p.] A general theorem about areas of pedals is proved in Gino Loria's book on Curves (2nd vol. of German edn., p. 316). References are there given to

Catalan, Mélanges Mathématiques (*Mém. Soc. Liége*, 2 Ser. xiii., 1886, p. 230).

Tsuruta, Counter Pedals (*Mess. Math.*, 2 Ser. xxiii., 1894).

Barisien, *Intermédiaire*, ii., 1896, 107-109 and 344, 345. A. R.

690. [K¹. 20. e.] *Some Theorems about the angles θ, ϕ, ψ, which any line makes with the sides of a triangle ABC.*



From the figure
$$\phi-\psi=A-180°.$$
$$\psi-\theta=B+180°.$$
$$\theta-\phi=C-180°.$$

By Trigonometry we can prove the following relations

$$\Sigma\sin(\phi-\psi)\sin\theta=0. \quad (1)$$
$$\Sigma\sin(\phi-\psi)\cos\theta=0. \quad (2)$$
$$\Sigma\sin(\phi-\psi)\cos(\phi+\psi)=0. \quad (3)$$
$$\Sigma\sin(\phi-\psi)\sin(\phi+\psi)=0. \quad (4)$$
$$\Sigma\sin(\phi-\psi)\cos(\phi-\psi)\sin 2\theta=0. \quad (5)$$
$$\Sigma\sin(\phi-\psi)\cos(\phi-\psi)\cos 2\theta=0. \quad (6)$$

(1) and (2) give
$$\Sigma\sin A\sin\theta=0.$$
$$\Sigma\cos A\cos\theta=0. \quad (A)$$

(5) gives
$$\Sigma\sin A\cos A\sin\theta\cos\theta=0.$$

But we also have:
$$\Sigma\sin(\phi-\psi)\cos(\phi-\psi)=\text{a constant}. \quad (7)$$
$$\Sigma\sin(\phi-\psi)\sin(\phi-\psi)=\quad\text{—} \quad (8)$$

(3) and (7) taken together give:
$$\Sigma\sin A\cos\phi\cos\psi=\text{a constant}.$$
$$\Sigma\sin A\sin\phi\sin\psi=\text{a constant}. \quad (B)$$

(4) and (8) taken together give:
$$\Sigma\sin A\sin\phi\cos\psi=\text{a constant}.$$
$$\Sigma\sin A\sin\psi\cos\phi=\text{a constant}.$$

(6) and (8) taken together give:
$$\Sigma\sin A\cos A\cos^2\theta=\text{a constant}.$$
$$\Sigma\sin A\cos A\sin^2\theta=\text{a constant}. \quad (C)$$

The constants may be obtained in every case by putting $\theta=0$.
Three of these results are useful in Trilinears.
Any line through $(\alpha\beta\gamma)$ is:
$$\frac{x-\alpha}{\sin\theta}=\frac{y-\beta}{\sin\phi}=\frac{z-\gamma}{\sin\psi}=s,$$
where s is the distance between (x, y, z) and (α, β, γ).

(A) gives the invariable relation between θ, ϕ, ψ, and (B) and (C) give the distance s in terms of (x, y, z), (α, β, γ) on substituting $\frac{x-\alpha}{s}$ for $\sin\theta$, etc.

University College, Southampton. H. V. LOWRY.

691. [V. 10.] *Enigma* (*Math. Gaz.* xi. p. 247).

The indices of the letters of the first word are 18, 14, 11, 9; of the second word 4, 5, 14; of the third word 7, 11, 14, 17, 9, 1. If we number up the letters of the alphabet, A=1, B=2, C=3, ... IJ=9,—the words emerge SOLI DEO GLORIA. R. F. D.

REVIEWS.

Commonsense of the Calculus. By G. W. BREWSTER. Pp. 62. 2s. net. 1923. (Clarendon Press.)

This little book will appeal to many boys and to some teachers, but if it comes into the hands of an exceptional boy, given to mathematical precision and exactness, it may make him unhappy. Some will think that the advice "to regard dx and δx as the same thing, provided δx is taken very small," savours rather of common carelessness than of commonsense, and will prefer the statement "$\frac{dy}{dx}$ is simply $\frac{\delta y}{\delta x}$ with the error squeezed out" to the later statement "you can get on quite well without bothering to distinguish between the two." The author appears to think that a studied vagueness is desirable in a preliminary survey, and is almost apologetic whenever he attempts to give exact meaning to his symbols. Is it beyond the reach of commonsense to distinguish between $\Sigma y\,\delta x$ and $\int y\,dx$? There is a certain charm about the author's style which may lead many to ask, like Oliver Twist, for more, and the "Historical Note" at the end of the book is especially interesting.

Calculus for Schools. By R. C. FAWDRY and C. V. DURELL. Pp. viii+300+xx. 6s. 6d. Or in two parts: Part I., 3s. 6d.; Part II., 4s. 1923. (Arnold.)

An excellent text-book which will at once take rank among the very best school books on this subject. It is clear and concise, yet convincing, straightforward and honest, well arranged, and fully illustrated with carefully graduated "examples which exhibit the practical applications of the subject and the variety of its ideas."

Part I. opens with a chapter designed to bring out the real meaning of "the equation of a curve." After that the subject-matter deals with Gradients; Differentiation; Maxima and Minima; Derivative as a Rate-Measurer; Functions of a Function; Indefinite Integrals; Definite Integrals; Approximate Evaluation of Definite Integrals; Applications of Integral Calculus to Kinematics, Work and Energy, Impulse and Momentum, Moments of Inertia, Fluid Pressure. The functions treated are the simple algebraic functions with which every schoolboy is familiar. Most teachers will agree with the authors in "the belief that pupils towards the end of their time at school gain more from encountering new ideas than strengthening their power of technique in formal Algebra and Trigonometry."

Part I. forms an admirable preparation, broad and deep, for the somewhat more formal Part II. which deals with General Methods of Differentiation; Derivatives of Trigonometrical Functions; Logarithmic and Exponential Functions; Differentials and General Integration; Applications to Geometry; Complex Number and the Hyperbolic Functions; Expansions in Series.

At the end of the book are Tables of Napierian Logarithms, of e^x, e^{-x}, $\cosh x$ and $\sinh x$, and 20 pages of Answers to the Examples.

The book has been carefully edited, so that very few errors are apparent: On page 123 the denominator of the last fraction should be $2a$ instead of a; on page 217, in Fig. 156, the wrong angle is marked θ; in the text of chapter ix. the solidus is omitted in "velocity u ft./sec." "acceleration a ft./sec.^{2}," etc., though it is inserted in the exercises that follow. These and a few other trifling misprints are the exceptions which prove the rule.

Plane and Spherical Trigonometry, with Stereographic Projections. By J. A. BULLARD and A. KIERNAN. Pp. v+230. n.p. 1923. (Heath.)

This book, the work of two professors at the United States Naval Academy, appears to be intended for the same type of student as Goodwin's well-known *Plane and Spherical Trigonometry*, which was written primarily to serve as an introduction to the study of Navigation and Nautical Astronomy for the junior officers under training in H.M. Fleet.

The first chapter is devoted to the theory and practical use of common

logarithms. In illustrating the use of a five-figure table the mantissa of log 31416 is repeatedly given as 0·47715. The use of negative characteristics is barred; thus $\bar{2}$ is written as 8 – 10, $\bar{1}2$ as 8 – 20, etc. Instructions, with examples, are given for the tabular arrangement of the logarithms in the various cases that may arise, reference being made to "given data" and "unknown data." The statement "b is negative and therefore log b is marked (–), 2 log b is not because b has been raised to an even power and as a result is positive" is not remarkable for clearness of expression.

Part I. deals with Plane Trigonometry, Part II. with Spherical Trigonometry, and the last two chapters are devoted to the very interesting subject of Stereographic Projections. They are based by permission on Professor Hendrickson's "Notes on Stereographic Projection," and the subject is developed in a simple manner up to the point at which it can be used for solving spherical triangles by geometrical construction and measurement. By the way, Messrs. Harrison and Baxandall in their *Practical Geometry and Graphics for Advanced Students* show (pp. 466 to 469) how the same can be done by the ordinary methods of Descriptive Geometry. W. J. DOBBS.

Cours de physique mathématique de la faculté des sciences. Compléments au tome III. Conciliation du véritable déterminisme mécanique avec l'existence de la vie et de la liberté morale. By J. BOUSSINESQ. Pp. xlviii+217. 30 fr. 1922. (Paris, Gauthier-Villars.)

A mathematical definition of *life* is a striking idea at least. This is what Prof. Boussinesq gives us. According to him, living organisms are those material systems whose differential equations and initial conditions admit of a singular solution as well as a particular integral. Whenever the system reaches a state common to the two solutions (for a particle in two dimensions, a point of contact of a trajectory with its envelope), the physical laws, of which the differential equations of motion are the statements, are not enough to determine along which path the system proceeds, and there must come into play some other cause or *principe directeur*, which is not a force, since it does not produce acceleration, but yet is necessary to determine the subsequent motion. Herein lies the possibility of free will even in a universe in which the laws of motion hold without any exception.

This interesting and undeniable proposition, whose main idea is traced back to Poisson, is just made quite clear, and then repeated in slightly varying forms at intolerable length. There is a good deal of much less convincing talk on the authority of *sens commun* and *sens intime*, and a certain confusion of language, if not of thought, on the distinction between zero and a physical quantity too small to be observed, and on the subject of non-Euclidean geometry, for which the author has no use at all.

An appendix contains a favourable review of an earlier version, but only the author's reply to a more critical notice.

The book deserves attention from those who care about the relation of mathematics to the rest of life. An excellent table of contents makes it easy to skip. H. P. H.

The "Akribos" Pocket Card of Four-Figure Logarithmic and other Tables. By W. H. BARKER. 2d. (Philip & Tacey.)

The "other" tables are: (1) Atomic weights of 40 elements, to the nearest integer; (2) Tensions of aqueous vapour for every degree from 7° C. to 26° C., in mm. of mercury, to one place of decimals; (3) Natural and logarithmic sines and tangents for every degree, to four places and without difference columns.

The table of logarithms has a system of dots by which entries in the difference column can be modified. The device does not secure four-figure accuracy in every entry, but it does reduce the proportion of entries in which the last figure is wrong, and in this way it strengthens the reliability of the third figure in an answer. On the other hand, one can easily make mistakes in utilising the dots, and although two of the vertical lines separating columns are in fact heavier than the others, the darkening is so slight that there is no effective protection against reading from the wrong column.

The card is handy in size, and the individual figures are clear.

Dimensional Analysis. By P. W. BRIDGMAN. Pp. viii+112. 25s. 1922. (Yale University Press; London, Humphrey Milford.)

To the mathematician this book seems an extravagance. The attaching of dimensions to a physical quantity is an important process, of which elementary explanations are invariably inadequate. But there are no mathematical difficulties, and Prof. Bridgman does not come seriously to grips with what for want of a better term we may call the philosophical difficulties. He deals well, however, with the "universal constants" of physics, and combats well the absurd but prevalent belief that dimensional analysis can be applied to phenomena about which no laws whatever are known.

We are perhaps prejudiced against the book for two reasons. (1) Considering that there are no diagrams, and that formulae compose only a fraction of the text and are of the very simplest kind, the price is preposterous. (2) The technique is concentrated into a theorem described as containing all the elements of the situation, but the proof is rendered worthless mathematically by an unusual fallacy: on p. 21 a conclusion as to the form of a function $f(\alpha, \beta, \gamma, \ldots)$ has been drawn from the assumption that $f(x\alpha, y\beta, z\gamma, \ldots)$ is to satisfy a certain condition for all values of the independent multipliers $x, y, z, \ldots$; on p. 39, $f(x\alpha, x\beta, x\gamma, \ldots)$ is to satisfy the same condition for all values of the single multiplier x, and the assertion is made that because this is a particular case of the previous assumption, therefore the previous conclusion can be drawn.

The value of the book is in the detailed discussion of a large number of examples; the hints given incidentally would be most useful to anyone tackling an actual problem for the first time. E. H. N.

The Mathematical Theory of Relativity. By A. KOPFF. Translated by H. LEVY. Pp. viii+214. 8s. 6d. net. 1923. (Methuen.)

Modern Electrical Theory. Supplementary Chapters. Chapter XVI.: **Relativity.** By N. R. CAMPBELL. Pp. viii+116. 7s. 6d. net. 1923. (Cambridge University Press.)

There are now several excellent text-books on the Theory of Relativity, and it is questionable whether publishers are wise in producing new ones. The two books considered here are rather overshadowed by Eddington's work, which has set a high standard.

Prof. Kopff is at his best in giving general introductory explanations, such as the opening chapter and the account of the *principle of equivalence*, though he does not mention that this principle is not always true. The mathematics follows closely that of Einstein and Weyl. The transformation formulae of the restricted theory are derived by a method given by Einstein in his little book for the general reader. Unfortunately there is something baffling about this method, which appears to deal with the ratio of two zeros, and at least two readers have jumped to the conclusion that Einstein's work is based on an elementary fallacy! The treatment of tensors is that of Weyl; it is not nearly as simple as that of Eddington. The book produces the impression that the author is a teacher of considerable ability, but that he has been handicapped by lack of access to the latest English work.

Dr. Campbell's book is written for experimental physicists, and he warns off those whose interests are primarily mathematical or philosophical. Accordingly, he does not give a connected chain of reasoning, but attempts to explain the salient points of the theory with as little mathematics as possible. The work is very unequal. Dr. Campbell has evidently tried hard to put the subject in a new way, but some of his changes are by no means improvements. This applies especially to the first portion of the book, which is far too verbose and not always clear. The account of the Michelson-Morley experiment contains an unfortunate slip (p. 27). On the other hand, the explanation of Minkowski's work is very helpful. The brief sketch of the generalised theory contains some shrewd criticisms which are worthy of serious consideration. However, we cannot accept the statement (p. 99) that the probable error of the 43″ advance of the perihelion of Mercury is about half the value. Dr. Campbell has erred in the opposite direction from most

books, which give 43″ as an accurate result. The total advance is known fairly closely, but as the masses of the other planets have not been determined very accurately, the amount of the perturbations caused by them is also uncertain. The residue for which the Newtonian theory does not account may be anything from 29″ to 43″. Dr. Campbell finds Einstein's argument concerning the spectral shift unconvincing. The book ends by refusing to express an opinion as to the truth or otherwise of the theory as a whole.

If this book is read in conjunction with some treatise on more orthodox lines it will throw some interesting sidelights on the subject.

H. T. H. PIAGGIO.

Un Théorème de Géométrie et ses applications. By G. CUNY. (Vuibert.)

These "notes," as the author modestly calls them, were prepared during lulls in trench warfare, and are now published with the idea that they may be useful to young students of mathematics, especially in showing the numerous and varied consequences which may be deduced from a general theorem. They seem likely to accomplish the author's aim, at any rate to such "jeunes gens" as have already made some progress in the geometry of Higher Plane Curves. Two fixed straight lines having been taken as axes of x and y and a third fixed one, ON (usually the bisector of the positive angle xOy), if an arbitrary fourth line $OM(y=mx)$ be taken, the anharmonic ratio of these four lines Ox, Oy, OM, ON is denoted by R_M, so that $R_M=m$ if ON is the bisector supposed. Having given some simple instances of the use of his notation in special cases of homographic pencils, the author explains the methods of inversion and reciprocation and their application. He then proceeds to the enunciation of his general theorem.

Let two straight lines Ox, Oy meet a curve (C_n) of order n respectively in n points A_1, $A_2 \dots A_n$, and n points B_1, $B_2 \dots B_n$.

Let a curve (C_p) of order p meet the curve (C_n) in np points α_1, $\alpha_2 \dots \alpha_{np}$ and the n straight lines A_1B_1, $A_2B_2 \dots A_nB_n$ in β_1, $\beta_2 \dots \beta_n$, then

$$R_{\alpha_1} \times R_{\alpha_2} \times \dots \times R_{\alpha_{np}} = R_{\beta_1} \times R_{\beta_2} \dots \times R_{\beta_{np}}.$$

Though the demonstration of this theorem presents no special difficulty, its generality involves the use of a very large number of symbols which conceals its simplicity, and students would probably find it useful to prove and apply the theorem to cases in which $n=2$ or 3, $p=1$ or 2 before considering it in its generality. The author admits, indeed, that the common value of the two products seems difficult to find in the general case. After several more or less general applications he devotes Chapter III., of 30 pages, to Conics.

The following, (i) and (ii), may serve as samples:

"(i) Considering the conics passing through four points A_1, A_2, B_1, B_2 situated two and two on Ox, Oy. Let M be a given point. Through M let a straight line be drawn meeting A_1B_1, A_2B_2 in M_1, M_2. If M' be taken on this line such that $R_MR_{M'}=R_{M_1}R_{M_2}$, M' lies on a conic through A_1, A_2, B_1, B_2, M (previously shown), and as there is only one point M' on the line M_1M_2M satisfying the above equation, it follows that a conic passing through five points is completely determined."

"(ii) If a conic C_1 passes through the origin O and has its asymptotes in the directions of Ox, Oy, and if OT be the tangent at O, and m and m' be the angular coefficients of the asymptotes of a conic C_2 cutting C_1 in A, B, C, D,

$$R_AR_BR_CR_D = R^2{}_T \times mm'.$$

Hence the product of the angular coefficients of the asymptotes of every such conic is constant, and therefore the directions of the asymptotes form an involution in which the axes Ox, Oy are self-correspondent.

"If the conic is a parabola,

$$m=m'=\pm\sqrt{R_AR_BR_CR_D}/R_{OT}.$$

Hence the directions of the axes of the two parabolas through A, B, C, D are harmonic conjugates with respect to Ox, Oy."

A final group of problems is left without demonstration for the reader to demonstrate by similar methods.

Ch. IV., 31 pages, is devoted to cubics (58-84) and quartics (85-90). (iii) and (iv) are interesting samples:

"(iii) If cubics pass through the vertices of a quadrilateral, through the point of intersection of pairs of opposite sides, through the intersection of the diagonals, and touch at one of the vertices, they osculate at this vertex.

"Let A, B, C, D be the vertices, F the intersection of the diagonals, E and G the intersection of pairs of opposite sides.

"Let the cubics have a common tangent AT at A, and let AB, AD be taken for the axes Ox, Oy. Let M be the ninth point common to these cubics, and let CMM' cut AT, BD, EG in γ, μ, μ'.

"By the general theorem,

$$R_\gamma R_\mu R_{\mu'} = R_G R_M R_{M'}.$$

Now R_G, R_M, R_γ, R_μ, $R_{\mu'}$ are constant for all the cubics. Hence $R_{M'}$ is also constant, and the cubics will pass through ten points, which is impossible unless M coincides with A, in which case they osculate at A."

"(iv) Let the two axes Ox, Oy cut a quartic in A_1, A_2, A_3, A_4; B_1, B_2, B_3, B_4. A conic passing through A_1A_2, B_1B_2 will meet the quartic in four other points. Let E, F be two of these, and let EF meet the quartic in two other points G, H, and meet A_1B_1, A_2B_2, A_3B_3, A_4B_4 in γ_1, γ_2, γ_3, γ_4.

"By the general theorem,

$$R_E R_F R_G R_H = R_{\gamma_1} R_{\gamma_2} R_{\gamma_3} R_{\gamma_4}.$$

"Since E, F are on a conic through A_1, A_2, B_1, B_2,

$$R_E R_F = R_{\gamma_1} R_{\gamma_2}. \quad \text{Hence } R_G R_H = R_{\gamma_3} R_{\gamma_4};$$

$$\therefore \ G \text{ and } H \text{ lie on a conic through } A_3, A_4, B_3, B_4."$$

In (iv) we have corrected an obvious misprint in the enunciation.

Ch. V. extends the general theorem to 3 dimensions and applies it to surfaces of the second degree.

While the theorems demonstrated by the author would sometimes yield quite readily to ordinary analysis, we agree with him that it is advantageous to have such an extensive collection treated by a uniform method. The fact that his angular coefficient of a line can be looked upon at will as the value of a certain anharmonic ratio enables him to add throughout to the theorems directly demonstrated by his method a large number of others obtained by inversion or reciprocation in which imaginaries are freely used.

The good paper and clear type are in these days something to be thankful for.

E. M. LANGLEY.

Scientific Thought. A Philosophical Analysis of some of its fundamental concepts in the light of recent physical developments. By C. D. BROAD. Pp. 555. 16s. (Kegan Paul.)

Mr. Broad has certain merits which make everything he writes useful in promoting clear ideas. He has a gift of lucid exposition of a point of view, whether his own or not; he has considerable sureness in appraising the weight of arguments which fail to be demonstrative; he can keep his head in the jungle of semi-conscious hypotheses through which most philosophising wanders; and he combines good sense with aptitude for delicate distinctions. He is not, and does not pretend to be, a profoundly original thinker; his forte is to expound the theories of pioneers better than they have done themselves, and to avoid being carried away by enthusiasm or repugnance into whole-hearted acceptance or rejection. These merits he displays in the present volume. There is little that is fundamentally new, but much that is judicious, and much that helps the reader to avoid confusions into which even the best writers are apt to fall.

The title of the book is somewhat wider than its subject. The only sciences that supply its material are physics and psychology; what is relevant of the former is only the theory of relativity, and of the latter only sensation and perception. Probably a physiologist would consider that some of the prob-

lems discussed demand more allusion to physiological data than Mr. Broad has thought necessary; but I do not doubt that he would be prepared with a fairly convincing answer to such a criticism. The subject of his book is Space, Time, and Matter. This, however, is the title of Weyl's book; and when I once asked a bookseller for Weyl's book he offered me first *Space, Time and Gravitation*, and then *Space, Time and Deity*. So it is no longer open to an author to call a book "Space, Time, and so-and-so." This is perhaps the reason for Mr. Broad's slightly misleading title.

The plan adopted—which is almost inevitable in a work of this kind—is to deal first with those parts of physics that are relevant to the philosophy of matter, and then with the problem of relating theoretical physics to the data of sense, which involves difficult questions in psychology and theory of knowledge. As an introduction, there is a justification of philosophy as a study. This proceeds on the assumption that the business of philosophy is to clear up the fundamental ideas and beliefs of the special sciences—*e.g.* to tell us what to mean by "matter" and what to assert when we profess to believe in causality. It cannot be denied that there is an important study which has these functions, but whether it should be called "philosophy" may be doubted. Cantor in the last generation showed us what to mean by "infinity" and "continuity"; Einstein in our own time has shown that a physical law must be expressible in tensor form. These were philosophical results according to Mr. Broad's definition, but Cantor and Einstein were not philosophers. The philosophers, in both cases, have done all that lay in their power to prevent the spread of the new clear ideas—by fallacious refutation in the first case and fallacious interpretation in the second. On a behaviourist basis, philosophy is to be defined as what a philosopher does. This is not (except in a few cases like Mr. Broad's) what Mr. Broad calls philosophy, which has been left mainly to mathematicians and physicists. I should myself, on behaviourist grounds, define "philosophy" as "the invention of fallacies to conceal our ignorance"; but that would compel me to deny that Mr. Broad is a philosopher.

Part I., dealing with space, time, and motion as required in physics, covers much familiar ground in a way intended to be intelligible to non-mathematicians. The reasons for accepting the special and general theories of relativity are set forth very clearly, and I doubt if the theories themselves can be better explained without mathematics. All this calls for no comment. One great merit of this Part is that Mr. Broad does justice to Dr. Whitehead's principle of extensive abstraction.

In discussing time, however, Mr. Broad makes certain concessions to Dr. McTaggart, perhaps out of respect for his predecessor at Trinity, which seem unnecessary and out of keeping with his general position. Dr. McTaggart distinguishes between before-and-after on the one hand and past-present-future on the other. With regard to the latter he professes to prove certain contradictions which are intended to show that the real cannot be in time at all. Mr. Broad does not accept this conclusion, but admits a good many of the arguments upon which it is based. The point of the difficulties is that events are not eternally past or eternally present or eternally future, but change in these respects. This suggests that past, present, and future are relations, but it is not at first sight easy to define the term to which they are to be related, since this term must be continually changing. Mr. Broad says: "Let us confine ourselves, for the sake of simplicity, to events that fall within the knowledge of a certain observer O .... Clearly we cannot simply define an event as present for O if O can perceive it or if it is contemporary with something that O can perceive. For we shall then have to define an event as past for O if O cannot perceive it but can either remember it or remember something contemporary with it. Now, of course, every event that falls within O's knowledge has these two incompatible relations to O; though, as we put it, it has them at different times." This assumes that the observer O persists through a longish period of time, whereas, according to the whole spirit of the philosophy of time which Mr. Broad accepts, whatever persists through more than a very short time should be regarded as composite. If an "observer" is really a series of "observations," Mr. Broad's argument fails. To say that an event is "present" is to say that it is contemporary with the momentary observer, whose duration is roughly that of a specious

present. There remains a logical point which needs to be cleared up: how are we to distinguish the momentary observer intended from other momentary observers, who equally, while they exist, use the word "now"?

This problem does not have to do solely with time. "Now" and "then" are analogous to "I" and "you," "here" and "there," in the fact that they have different meanings on different occasions, although on each occasion their meaning is definite. "This" is another such word, perhaps the most fundamental. We may take "this" to be an ambiguous proper name, but its ambiguity is of a peculiar sort: it names different things on different occasions, but names a definite thing on each single occasion. "This" *names* the object in the focus of the speaker's attention, but does not *mean* "the object in the focus of the speaker's attention," for, if it did, it would be always applicable to everything that some speaker at some time attends to. We may define "present" as "simultaneous with this" and "past" as "earlier than this" and "future" as "later than this." Thus past, present, and future are relations to a relatum designated by an ambiguous proper name.

Mr. Broad's view is that the future is in some way less real than the past and the present. He says: "Such a theory as this [his own] accepts the reality of the present and the past, but holds that the future is simply nothing at all." To my mind, this is a mere confusion due to the unfortunate fact that verbs have tenses. We may say "the future *is* nothing," and equally "the present *will be* nothing and *was* nothing." All these statements are mere tautologies, equivalent to: "The statement that a thing exists at time t is not equivalent to the statement that it exists at time t', unless $t=t'$." In this latter statement, "exists" is used without tense, whereas in Mr. Broad's statement "is" is used in the present tense, because the past, through memory, is supposed to be still somehow part of the sum-total of reality. I do not think it is humanly possible to arrive at a just philosophy of time except by employing verbs in a sense divorced from tense. We may then say that past, present, and future are [in a sense without tense] all equally real; what is false is to say that the future "is" when "is" means "is now," as it normally does. I think there is a similar confusion in Mr. Broad's contention that the time-series essentially has sense from earlier to later, rather than *vice versa*. I think this amounts merely to the empirical fact that memory only works one way. There is no reason, except brute fact, why we should not remember the future as well as the past; but if we did, our philosophy of time, free will, morals, punishment, etc., would be radically different.

Part II. of Mr. Broad's book, "On the sensational and perceptual basis of our scientific concepts," is more important than Part I. It is obvious that physics rests upon empirical evidence, *i.e.* evidence derived from objects of sense—from what we experience in seeing and hearing, and so on; but it is by no means obvious how we can build a logical bridge from such data to the abstract and imperceptible objects of our mathematical formulae. This is a subject with which Dr. Whitehead has dealt, and Mr. Broad in the main accepts Dr. Whitehead's views. But he expands and clarifies them, adding a great deal of admirable detail. His discussions of visual space, the movement-continuum, somatic sensations, etc., and the parts they have respectively played in leading to the conceptions of common sense first and then scientific physics, can hardly be summarised, but should be read by all who are interested in the epistemological status of such objects as electrons. The type of theory advocated by Mr. Broad is by this time familiar. When a number of people look at a penny, most of them see, not circles, but ellipses of various sizes and eccentricities. Unless we admit that these are all in some sense real, we destroy the empirical basis of physics; but if we do admit them, our theories of space and of physical objects must be very complicated. By means of mathematical logic, it is possible, out of the innumerable "appearances" of the penny, to construct the sort of space and the sort of physical object that the mathematical physicist requires; but the result is that things which he takes as simple become elaborate logical structures. This does not matter to the physicist, to whose work it is irrelevant; but it is interesting to the philosopher, who wants to know in what sense (if any) matter exists. It is also interesting to the mathematical logician, since it shows that his

comparatively modern branch of mathematics gives the indispensable instrument for answering certain questions which most philosophers hoped were unanswerable.

The book ends with a chapter on "The physiological conditions of sensations, and the ontological status of sensa," which is concerned chiefly with refuting the argument that our sensations are so dependent upon physiological conditions as to be unable to give us any reliable data concerning the outer world. There is of course a logical retort to this argument, namely that our knowledge of physiology is obtained by just those means which the argument seeks to discredit, and that therefore physiology itself collapses if the argument is sound. A retort of this kind is, however, not satisfying; moreover, taken alone, its only effect should be physical scepticism. Mr. Broad accordingly gives other answers. I think that here, as elsewhere, he falls into needless complications by distinguishing between the act of sensing and the object sensed, which he calls a *sensum*; but these complications do not involve any very vital element in his theory. His theory appears to me (though not to him) to be almost the same as that which I have advocated on the same subject, except for the important difference that he admits a more radical distinction between the mental and the physical than I regard as necessary. In a theory of the physical world, however, this need not have much relevance. The question at issue is as to what goes on in places where there is no observer. We know what happens when we see a star, and we believe that a process called light-waves takes place throughout the intervening space. On the face of it, a sensation, as known in psychology, is utterly different from a light-wave, as known in physics—so different that the causal relation between them seems very strange. It seemed worth while to show that a light-wave *may* be composed of things which are analogous to sensations as regards spatio-temporal extent and position, and that such things may, as they approach the observer's brain, become gradually more and more like his sensation, so that there is no breach of continuity when we come to the last link in the causal chain. The avoidance of such a breach, when possible, is always recognised as a merit in a scientific theory; and to secure this merit is one of the objects of the views which Mr. Broad advocates.

Mr. Broad objects to me that if, as I suppose, our brains and nerves have the same kind of effect as coloured glasses might have in modifying the local occurrence connected with an object in the way in which, in physics, light-waves are connected with the star from which they emanate, then, since we can never get away from our nerves and brain, we can know nothing about what goes on outside them. "We therefore do not really know that sensa can exist at all apart from brains and nervous systems. And, even if we decide to postulate sensa of *some* kind in places and times where there are no brains and nervous systems, we cannot have the slightest idea what intrinsic sensible qualities such sensa will have. . . . To call them *sensa*, under these circumstances, seems rather misleading; for it is liable to disguise the purely hypothetical character of these events, and to suggest that we know a good deal about their intrinsic qualities. Really we know nothing about the events which happen at intermediate times and places between the opening of a shutter and our sensing of a flash, except that they obey Maxwell's Equations."

To this I should reply, first, that I never have called such things "sensa." At one time I called them "sensibilia," for the special purpose of distinguishing them from sensa; then I adopted the more colourless word "particulars" in order to weaken still further the analogy with sensa. Secondly, although we cannot view the world without the medium of nerves and brain, we can view it in different conditions of nerves and brain, and we find that it looks different as the conditions vary. This makes it difficult to avoid the conclusion that our nerves and brain colour what we call the appearances of physical objects. Thirdly, I admit to the full that we do not know the intrinsic qualities of unperceived particulars; strictly speaking, we do not know even that they obey Maxwell's Equations, since we do not know that they exist. No logically cogent argument proves that anything unperceived exists; we assume this, first instinctively, then for the sake of continuity, because a merely sensational world would be intolerably jagged. The assumption of continuity in the unperceived processes leads to Maxwell's Equations, but does not, of

itself, provide continuity with perceived processes. If we can provide this too, without sacrificing Maxwell's Equations, we have done more to satisfy the postulate of continuity which leads us to demand a physical world. The argument is not, and does not pretend to be, demonstrative, but it is of the same order as the argument for a physical world, and strengthens this argument by allowing fuller scope for continuity. In all this, however, I do not feel that I am in any substantial disagreement with Mr. Broad. I should not have wasted space on a personal matter, except for the hope that the discussion might help to clarify the questions concerned.

Mr. Broad's book is addressed rather to the philosophical student who wishes to understand the principles of science than to the man of science who is interested in philosophy. There is something of a gulf between its two parts, but that, I think, is unavoidable in a philosophical discussion of matter. I hope many philosophers will read the book and will understand that the theory of relativity is what Mr. Broad says it is, and not what some of the more venturesome of the philosophers try to persuade their colleagues that it is. The book fulfils its purpose admirably, and should do much to clarify the ideas of philosophical students as to what is meant by "matter" and "space-time."

BERTRAND RUSSELL.

Newton. By GINO LORIA. Pp. 69. L. 2. 1920. (Formiggini, Rome.)

This monograph from the indefatigable pen of Prof. Loria forms the 52nd number of a series called "Profili." The choice of profiles is catholic enough, the first two being those of Botticelli and Darwin, and others range from Galileo to Charles Dickens, and Rossini to Julian the apostate. The main source of the author's inspiration is probably Brewster's *Life*. He has often lamented that while a Cayley and a Sylvester have had their collected editions, we still have to await the modern edition of the works, or even a definitive *Life* of one of the greatest of Englishmen. But from the usual material Prof. Loria has compiled within small compass an account of Newton's work and career, which is sympathetic, trustworthy and effective, and is just what is needed for non-mathematical readers.

The Nebular Hypothesis and Modern Cosmogony (The Halley Lecture of May 23, 1922). By J. H. JEANS. Pp. 31+4 plates. 2s. 6d. net. 1923. (Clarendon Press.)

The volume on the Problems of Cosmogony, published by Dr. Jeans in 1920, was based upon theoretical investigations into the behaviour of rotating masses. The Halley Lecture before us, after a brief account of the work of Laplace and the eighteenth century, discusses the present position of the nebular hypothesis as affected by modern mathematical theory and observational tests. The nebulae, in number about half a million, appear to lie on a continuous evolutionary chain. Dr. Jeans devotes the greater part of his lecture to the consideration of these, "in the belief that the study of the formations on this chain provides us with the fullest answer at present available to the problem propounded by Laplace." The sequence of configurations is reviewed on pp. 25-26. The rest of the lecture is devoted to a summary of the tidal theory of the genesis of the solar system. It is "just within the bounds of possibility, although quite, I think, outside the bounds of probability, that our system is unique. . . . Our sun may be the only one attended by satellites . . . our earth may be the only body in the whole universe which is capable of supporting life." Finally, "if (the cosmogonist) makes a positive assertion, the only one he is entitled to make is that in cosmogony we know nothing at all for certain."

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188. She shall have clothes, but not made by geometry.—Beaumont and Fletcher's *Elder Brother*, II. ii.

189. The only further news . . . is the dismissal from Trinity of . . . for rowing, and throwing Peacock's books out of the window.—Monckton Milnes (Lord Houghton), Dec. 1829.

THE LIBRARY.

160 Castle Hill, Reading.

It is proposed to bind for the Library a set of Lists of Members of the A.I.G.T. and the M.A. At first a list formed part of the Annual Report of the A.I.G.T., but from the 15th Report (Jan. 1889) onwards, none was included. We have a complete set of M.A. Rolls, which were issued before the war regularly every other year, from 1898. But of the separate A.I.G.T. Rolls we have only three, one undated, prepared apparently late in 1889, and two dated, 1892 and 1893. If anybody can supply missing Rolls to be incorporated, the Council will be very grateful.

ADDITIONS.

The Librarian reports the following gifts:

From Prof. R. W. Genese:

(1) A collection of examination papers and guides.

(2) School text-books on various mathematical subjects by the following authors: T. S. Aldis, H. M. Andrew and F. J. Pirani, S. Barnard and J. M. Child, C. Davison, R. Deakin, W. C. Fletcher, C. H. French and G. Osborn (two), B. R. Green, H. S. Hall, C. Hoare, H. S. Jones, A. Knox, J. B. Lock and J. M. Child, J. B. Lock and V. M. Turnbull, P. Lyddon-Roberts and E. E. Denney, J. Mackean, D. B. Mair, J. W. Marshall and C. O. Tuckey, G. W. Palmer, C. Pendlebury (two), B. Smith, H. M. Taylor, W. Weeks, J. M. Wilson (two).

(3) The following books, which can not be described as school text-books:

A. C. Beaton	Quantities and Measurements - - - - - - *A practical builder's guide to estimating.*	1891
J. A. Galbraith and S. Haughton	Manual of Arithmetic - - - - - - - -	1858
C. S. Jackson and R. M. Milne	A First Statics - - - - - - - - -	1907
G. Salmon	Conic Sections - - - - - - - - -	1850
R. Wolf	Taschenbuch für Mathematik usw. - - - -	1856
G. C. Young and W. H. Young	The First Book of Geometry - - - - - - -	1905

(4) Mathematical tables: Chambers (1860), Clive (1906, compiled by A. G. Cracknell), W. Hall (1905), E. Mougin [1913], W. E. Paterson [1911], O. Schlömilch (1878), C. J. Woodward (1909).

(5) The following apparatus:

Pantographe américaine, made by J. Conte, Paris.

Peaucellier's cell.

Stereometrie. (*A German set of cards, cut out and scored, for the construction of polyhedra; the figures are like some of those in Cowley's* Appendix to Euclid, *but the cards are loose and details of mensuration are printed on them.*)

Stereoscopic Views of Solid Geometry Figures, with references to Wells's *Essentials of Solid Geometry* (Boston, U.S.A., D. C. Heath & Co., 1899).

Réglettes Financières, Multiplicatrices, Multisectrices, et Népériennes, inventées par Henri Genaille et perfectionnées par Edouard Lucas (Paris, E. Belin, 1885. *Four sets of calculating rods: we may grant the originality of the first three, but the publisher's claim regarding the fourth can hardly have been endorsed by a mathematician of M. Lucas's rank.*)

From Mr. W. J. Greenstreet (second list):

L. Euler	Institutiones Calculi Differentialis - - - - -	1755
	Introductio in Analysin Infinitorum (2 vols.) - - -	1748
G. Fazzari	Aritmetica (2 editions) - - - - - -	1916, 1918
P. de Fermat	Œuvres (3 vols.) - - - - - -	1891, 1894, 1896
L. B. Francœur	Complete Course of Pure Mathematics. Trans. (from French into English) by R. Blakelock (2 vols.)	1829, 1830
A. Fuhrmann	Aufgaben aus der Analytischen Mechanik. Part 1 -	1904

(*Will any member supply the completing Part 2?*)

J. O. HALLIWELL	Rara Mathematica	1839
C. HERMITE	Cours d'Analyse. Part 1 (all published)	1873
J. F. W. HERSCHEL	Examples of . . . . Finite Differences	1820
W. HOOPER	Rational Recreations (4 vols.)	1794
C. HUTTON	Mathematical Tables	1830
	Mensuration	1812
J. HYMERS	Differential Equations and Calculus of Finite Differences	1839
C. JORDAN	Cours d'Analyse. 2nd ed. Vol. 1	1893
	(*Another opportunity to members; this edition was in 2 vols.*)	
P. KELLAND	Principles of Demonstrative Mathematics	1843
J. L. LAGRANGE	Œuvres. Vols. 1-13	1867-1889
	(*The Council has purchased Vol. 14 to complete this set.*)	
	Résolution des Equations Numériques	1826
	Théorie des Fonctions Analytiques	1813
C. A. LAISANT	Equipollences	1887
P. M. H. LAURENT	Traité d'Analyse. Vol. 4: Théorie des Fonctions Algébriques et de leurs Intégrales	1890
	(*The complete work is in 7 vols.*)	
	Vol. 1, 1885: 2, 1887: 3, 1888: 5, 6, 1890: 7, 1891.	
C. MACLAURIN	Algebra	1748
M. MARIE	Réalisation et Usage des Formes Imaginaires en Géométrie	1891
"THE MATHEMATICIAN."		1745-1751
	(*A treatise on Conics, issued in six parts.*)	
H. MULLER und J. PLATH	Lehrbuch der Mathematik	1906
(I. NEWTON)	Arithmetica Universalis	1707
I. NEWTON	Universal Arithmetick. Trans. (from Latin into English) by J. Ralphson. Ed. by T. Wilder. Part 1	1769
	(*There is a second part, much to be desired.*)	
W. F. OSGOOD	Lehrbuch der Funktionentheorie. Vol. 1	1907
	(*Vol. 2 is wanted.*)	
G. PEACOCK	Examples of the . . . . Calculus	1820
J. PETVIN	Letters concerning Mind, etc.	1750
E. PICARD	Traité d'Analyse. Vol. 1	1891
	(*This edition requires 3 more vols. to complete it.*)	
	Wissen der Gegenwart in Mathematik und Naturwissenschaft. Trans. (from French into German) by F. and L. Lindemann	1913
J. PLUCKER	System der Analytischen Geometrie	1835
	System der Geometrie der Raumes	1846
	Theorie der algebraischen Curven	1839
H. POINCARE	Wert der Wissenschaft. Trans. (from French into German) by E. and H. Weber	1910
	Wissenschaft und Methode. Trans. (from French into German) by F. and L. Lindemann	1914
G. POSTELLUS	De Cosmographica Disciplina	1636
B. PRICE	Infinitesimal Calculus. Vols. 2 (1st ed.), 3-4 (2nd ed.)	1854, 1868, 1889
	(*Vol. 1, 1st ed., is already in the Library.*)	
J. A. SERRET	Algèbre Supérieure (2 vols.)	1885
	Calcul Différentiel et Intégral (2 vols.)	1868
	Differential- und Integral-Rechnung. Trans. (from French into German) by A. Harnack. Vol. 3	1904
	(*As a treatise on differential equations and the calculus of variations, this volume is complete in itself, but the preceding volumes would be none the less welcome.*)	
T. SIMPSON	Algebra	1767
	Select Exercises for Young Proficients in the Mathematicks	1752
R. SIMSON	Conic Sections. Trans. (from Latin into English)	1804
G. G. STOKES	Memoir and Scientific Correspondence (2 vols.)	1907
F. G. TEIXEIRA	Obras. Vols. 1, 4, 5	1904, 1908, 1909
	(*Most of the work is in French. Vols. 4 and 5 together constitute a* Traité des Courbes Spéciales Remarquables Planes et Gauches.)	

H. WEBER	Algèbre Supérieure. Trans. (from German into French) by J. Griess. Vol. 1 - - - - - - -	1898
	(This volume corresponds to the first half of the original Lehrbuch; it is complete in itself, but there is a sequel.)	
J. WOLSTENHOLME	Differential and Integral Calculus - - - - -	1874

Mrs. JOURDAIN has presented a considerable number of offprints of papers by her late husband. Members who are interested in Mr. P. E. B. Jourdain's work may have an offprint of any particular paper at cost of postage.

From Mr. E. M. LANGLEY:

A.I.G.T. Reports, 1871-1886, bound in one volume.

This gift is specially welcome, as the bound set already in the Library lacked the Second Report, dated Jan. 1872. We have two other sets from which only this one number is missing, which would be invaluable for purposes of exchange if members could complete them.

Also 45 back numbers of the *Gazette*.

From Rev. J. J. MILNE:

(1) The original edition of a classic, and a contemporary translation of the same work:

C. F. GAUSS	Disquisitiones Arithmeticae - - - - - - -	1801
	Recherches Arithmétiques. Trans. (from Latin to French) by A. C. M. Poullet-Delisle - - - - - -	1807

A COLLECTION OF WORKS ON GEOMETRICAL CONICS.

(2) With a few exceptions, the titles are variations of one phrase, and the books can be identified by authors and dates as follows:

B. BRIDGE - -	1811	
H. G. DAY - -	1868	
W. H. DREW - -	1869	
H. HAMILTON - -	1758 *Latin*	
	1773 *English trans.*	
S. H. HASLAM and J. EDWARDS -	1881	
R. JACK - -	1742	
N. DE LA CHAPELLE	1750 *French*	
F. S. MACAULAY -	1895	
J. MILNES - -	1712	
T. NEWTON - -	1794	
D. M. PEACOCK -	1809, 1817 *1st and 2nd eds.*	
G. RICHARDSON -	1888	
A. ROBERTSON -	1792 *Latin, with historical appendix*	
	1818 *English, a distinct work*	
R. SIMSON - -	1735, 1750 *Latin, 1st and 2nd eds.*	
R. STEELL - -	1723	
C. TAYLOR - -	1863, 1889 *These are distinct works*	
L. TREVIGAR -	1731	
S. VINCE - -	1781, 1810 *1st and 4th eds.*	
T. WADDINGHAM -	1857	
W. WALLACE -	1837	
W. WHEWELL -	1849	
H. G. WILLIS -	1883	

The exceptional titles may be abreviated thus:

W. H. DREW	Solutions to Problems - - - - - - -	1868
T. GASKIN	Construction of a Conic Section - - - - -	1852
H. LATHAM	Geometrical Problems in . . . Conic Sections - - -	1848

Also there are a reprint of H. M. Taylor's article on Geometrical Conics in the *Encyclopaedia Britannica*, and three anonymous works:

The Elements of the Conic Sections with the Sections of the Conoids - - - -	1818
The same, 3rd ed. -	1826
A Geometrical System of Conic Sections. For the use of mathematical students at the Liverpool Royal Institution - - - - - - - - - - - - - -	1822

For the purpose of Cataloguing, the Librarian would be grateful for any information bearing on the authorship of these books.

It should be recalled that the Library already owes to Mr. Milne's generosity superb copies of de la Hire, 1685, and of Halley's edition of *Apollonius*, 1710, as well as many books on subjects other than geometrical conics.

Sir THOS. MUIR, to whom the Library is already indebted for the first three volumes of his monumental work, *The Theory of Determinants in the Historical Order of Development*, has sent the fourth volume, *The Period* 1880 *to* 1900, published this year, and has enriched the copy by the inscription, signed and dated, "To the Library of the Mathematical Association with best wishes for the Library's progress."

To the collection of pamphlets the following have been added, the first by Prof. Archibald, the second by Canon Wilson:

Q. VETTER	The Development of Mathematics in Bohemia - - Trans. from Italian and edited with notes by R. C. Archibald. Offprint from *Amer. Math. Monthly*, vol. xxx. p. 47.	1923
J. M. WILSON	Mathematical Teaching, especially of Geometry - - A lecture given by request in the Royal High School, Edinburgh.	1870

GAPS.

The following is a list of those works issued in more volumes than one, of which the Library holds an incomplete set. It is published in the hope that there may be members in a position to make good some of the defects.

WORK	POSSESSED	WANTED
J. BLAIKIE and W. THOMSON, Geometrical Deductions -	Book 1	?
O. BOLZA, Variationsrechnung, 1909 - - - -	Parts 2, 3	Part 1
G. BOOLE, Differential Equations, 1872 - - -	Treatise	Supp. Vol.
— BRESSE, Cours de Mécanique Appliquée - - -	Vol. 2	Vol. 1
T. BURNET, Sacred Theory of the Earth - - - (*This is an 8vo edition, undated but after* 1699.)	Vol. 1	Vol. 2
F. ENRIQUES, Fragen der Elementargeometrie, German trans., 1907 - - - - - - - - -	Vol. 2	Vol. 1
L. EULER, Elements of Algebra, English trans., 1810 -	Vol. 1	Vol. 2
G. FAZZARI, Elementi di Aritmetica - - - -	Part 1	?
A. FUHRMANN, Aufgaben aus der Analytischen Mechanik, 1904 - - - - - - - - -	Part 1	Part 2
L. HENSLEY, Figures made Easy, 1872 - - -	Answers to Ex.	Book
G. A. HIRN, Théorie Mécanique de la Chaleur, 1865 -	Vol. 1	Vol. 2
J. A. HULSSE, Maschinen-Encyclopädie, 1841→ - -	Vols. 1, 2	Vols. 3→
C. HUTTON, Course of Mathematics, 3rd ed. - -	Vol. 3	Vols. 1, 2
H. A. JAMIESON, Oxford Elementary Arithmetics -	{ Pupils' 2, 3, 4 / Teachers' 1, 5, 6	?
C. JORDAN, Cours d'Analyse, 2nd ed., 1893 - - -	Vol. 1	Vol. 2
P. M. H. LAURENT, Traité d'Algèbre, 1887 - - -	Vol. 2	Vol. 1
Traité d'Analyse, 1885-1891 - -	Vol. 4	Vols, 1-3, 5-7
C. F. A. LEROY, Géométrie Descriptive - - -	Vol. 2 (Plates)	Vol. 1 (Text)
E. LIONNET, Arithmétique, 1857 - - - - -	Complément	Eléments
J. B. LOCK, Mechanics for Beginners, 1891 - - -	Part 1	Part 2
L. LORENZ, Œuvres, 1898 - - - - - - -	{ Vol. 1, Part 2 / Vol. 2, Part 1	Vol. 1, Part 1 / Vol. 2, Part 2
D. A. LOW, Practical Solid or Descriptive Geometry, 1889 - - - - - - - - -	Part 1	Part 2
W. F. MEYER, Differential- und Integral-Rechnung, 1907 - - - - - - - - - -	Vol. 1 (Diff.)	Vol. 2 (Int.)
J. J. MILNE and R. F. DAVIS, Geometrical Conics, 1890, 1894 - - - - - - - - - - (*The Library does contain the one-volume form of this work.*)	Part 1	Part 2
J. H. T. MULLER, Lehrbuch der Mathematik - -	Vol. 2, Parts 2, 3	{ Vol. 1 / Vol. 2, Part 1
I. NEWTON, Opera, ed. Horsley, 1779 - - - -	Vols. 1, 2	Vols. 3-5
Universal Arithmetick. Ralphson's trans., ed. Wilder, 1769 - - - - -	Part 1	Part 2
W. F. OSGOOD, Lehrbuch der Funktionentheorie, 1907 -	Vol. 1	Vol. 2

WORK	POSSESSED	WANTED
E. PASCAL, Repertorium der Höheren Mathematik, 1902	Vol. 2	Vol. 1
G. PEANO, Formulaire de l'an 1902-3 - - - - -	Parts 2-3	Part 1
J. PLUCKER, Analytisch-Geometrische Entwicklungen, 1831 - - - - - - - - - -	Vol. 2	Vol. 1
W. J. S' GRAVESANDE, Mathematical Elements of Natural Philosophy. Désagulier's trans., 3rd ed., 1726 - (*This is an 8vo edition; the Library has the 6th ed., 1747, 4to, complete.*)	Vol. 1	Vol. 2
H. M. TAYLOR, Pitt Press Euclid, 1889-1895 - -	I-II, III-IV	V-VI, XI-XII
W. W. TAYLOR, Solutions to Exercises in Taylor's Euclid, 1897 - - - - - - - - - -	VI-XI	I-IV
I. WARREN, Mechanics, 1889 - - - - - - (*The copy of Part* 1 *is imperfect, lacking pp.* 129-130.)	Part 1	Part 2
H. G. WILLIS, Elementary Modern Geometry, 1905 -	Part 1	Part 2
S. H. WINTER, Elementary Geometrical Drawing, 1883	Part 1	Part 2
R. WOODHOUSE, Astronomy, 1821-3, 1818 - - -	Vol. 1	Vol. 2
W. G. WOOLLCOMBE, Practical Work in Physics, 1899 -	Part 2	Parts 1, 3, 4

A REQUEST.

A member wishes to borrow Bolzano's *Paradoxien der Unendlichen*. Anyone able and willing to lend the book is asked to communicate with the Librarian.

NOTICE.

The Editor acknowledges receipt of a Note on criteria of divisibility, signed *Tria*, but accompanied by no name or address. The postmark seems to be Norwich.

FOR SALE.

Tisserand's Mécanique Céleste, in good condition (unused). Complete in four volumes. Published £6 6s. Price £2 16s. Apply C. V. GRANT, Kirmington, Ulceby, Lincs.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
AT THE UNIVERSITY PRESS, GLASGOW

BOOKS RECEIVED, CONTENTS OF JOURNALS, ETC.

October, 1923.

Cours de Mécanique Celéste. Vol. I. By M. H. ANDOYER. Pp. 140. 50 fr. 1923. (Gauthier-Villars.)

Academia pro Interlingua. Historia. Circulare. No. 1. (Prof. G. Peano. University of Turin.)

A General Text-Book of Elementary Algebra. By E. H. CHAPMAN. Part II. Pp. 145-256 + xxxviii. 3s. 3d. net. 1922. (Blackie.)

Plane and Spherical Trigonometry, with Stereographic Projections. By J. A. BULLARD and A. KIERNAN. Pp. iv + 230. n.p. 1922. (Heath.)

Researches in Geometry. By Drs. P. S. G. DUBASH, CHROM, and SAN. Pp. 11. 8 annas. n.d. (Taraporewalla & Sons, Hornby Road, Fort, Bombay.)

Constructive Arithmetical Exercises. Part I. with Answers. By R. W. M. GIBBS. Pp. x + 244 + 44. 3s. 6d. net. 1923. (Blackie.)

Scientific Thought. A Philosophical Analysis of some of its Fundamental Concepts in the Light of Recent Physical Developments. By C. D. BROAD. Pp. 555. 16s. net. 1923. (Kegan Paul.)

Contribucion al Estudio de las Ciencias físicas y matemáticas. Universidad Nacional de La Plata. June 1922.

The Mathematical Sciences in the Latin Colonies of America. By F. CAJORI. Pp. 193-203. Reprint from *The Scientific Monthly.* Feb. 1923.

Modern Electrical Theory. Supplementary Chapters. Chap. XVI. *Relativity.* By N. R. CAMPBELL. Pp. viii + 116. 7s. 6d. net. 1923. (Camb. Univ. Press.)

The Mathematical Theory of Relativity. By A. KOPP. Pp. viii + 214. 8s. 6d. net. 1923. (Methuen.)

The 'Rapid' Decimal Calculator and Universal Reckoner. By J. GALL INGLIS. Size $4\frac{5}{8} \times 3\frac{1}{4}$. 2s. 6d. net. 1923. (Gall & Inglis.)

Matriculation Algebra. By C. V. DURELL and G. W. PALMER. Pp. viii + 144 + 6 + xvi. 3s. 1923. (Bell.)

The "Akribos" Pocket Card of Four Figure Logarithmic and other Tables. By W. H. BARKER. Pp. 4. 2d. net. 1923. (Philip & Tacey.)

Calculus for Schools. By R. C. FAWDRY and C. V. DURELL. Pp. viii + 300 + xx. 6s. 6d. Part I. 3s. 6d. Part II. 4s. 1923. (Arnold.)

Fundamental Congruence Solutions, giving one root (y) of every Congruence of $y^{\xi} \equiv +1 \pmod{p \text{ and } p^{\kappa}}$ *for all primes and prime-powers* $< 10{,}000$. By A. CUNNINGHAM and (the late) T. C. CREAK. Pp. xviii + 92. n.p. 1923. (Hodgson.)

The Dissection of Rectilineal Figures. By W. H. MACAULAY. Pp. 53-56. Reprinted from the *Messenger of Mathematics.* Aug. 1922.

George Bruce Halstead. By F. CAJORI. Pp. 338-340. Reprint from *Amer. Math. Monthly.* Oct. 1922.

Transcendent Algebra. Ideografie Matematical. Experiment de un lingue filosofic. 1921. (Reval, Estonia.)

Kosmoglott. Organ del Soc. Kosmoglott. (Reval, Nikitinstrad 10.)

Transformations of Surfaces. By L. P. EISENHART. Pp. ix + 380. 20s. net. 1923. (Oxford University Press.)

Introduction to Practical Mathematics. By V. S. BRYANT. Pp. 95. 2s. 6d. net. 1923. (Clarendon Press.)

Descartes e la Teoria dei numeri. By GINO LORIA. Pp. 11. Reprint from *Fasc.* I. 1923. *Bollettino di Matematica.*

Applicazioni geometriche di una formula di F. Siacci. By GINO LORIA. Pp. 11. Reprint from *Bolletino dell' Unione Matematica Italiana.* June 1923.

American Journal of Mathematics. (Wesley & Son, London.)

XLIV. 4.

On the Kernel of the Stelljes Integral corresponding to a completely continuous Transformation. Pp. 237-264. C. A. FISCHER. *Equivalence and Reduction of Pairs of Hermitian Forms.* Pp. 247-260. M. I. LOGSDON. *Plane Cubics with a given Quadrangle of Inflexions.* Pp. 261-278. B. M. TURNER. *Normal Ternary and Continued Fraction Expansion for the Cube Roots of Integers.* Pp. 279-296. P. H. DANS. *Concerning compact Kürschák Fields.* Pp. 297-316. V. D. GOKHALE.

Jan. 1923.

On the number of Solutions in Positive Integers of the Equation $yz+zx+xy=0$. Pp. 1-4. L. J. MORDELL. *A Closed Set of Normal Orthogonal Functions.* Pp. 5-24. J. L. WALSH. *Congruences determined by a given Surface.* Pp. 25-42. C. KENDALL. *Linear Partial Differential Equations with a Continuous Infinitude of Variables.* Pp. 42-53. I. A. BARNETT. *On the Ordering of the Terms of Polars and Transvectants of Binary Forms.* Pp. 54-71. L. ISSERLIS.

The American Mathematical Monthly. (Lancaster, Pa.)

Nov.-Dec. 1922.

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